

Cohn localization of finite group rings.

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Abstract. The purpose of this paper is to give a complete description of the Cohn localization of the augmentation map $\mathbf{Z}[G] \rightarrow \mathbf{Z}$ when G is any finite group.

Keywords: Non-commutative localization, category of chain complexes.

Introduction. Let W be a set of matrices with entries in a ring A . The Cohn localization of A with respect to W is a ring obtained from A by formally inverting all matrices in W . This ring Λ is equipped with a ring homomorphism $A \rightarrow \Lambda$ and every matrix in W becomes invertible in Λ . Moreover Λ is universal with respect to this property.

A particular example of such a localization may be obtained from a ring homomorphism $A \rightarrow B$. In this case the Cohn localization of A with respect to the set of matrices with entries in A becoming invertible in B will be denoted by $L(A \rightarrow B)$. Because of the universal property, the morphism $A \rightarrow B$ factors through $L(A \rightarrow B)$.

If A is commutative, a Cohn localization Λ of A is a classical localization $S^{-1}A$, where S is a multiplicative set in A and Λ is a flat A -module. In the general case Λ is much more complicated, in particular if $A \rightarrow B$ is the augmentation map of a group ring $\mathbf{Z}[G]$. In this case, the ring $\Lambda = L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ is known if G is commutative or free [FV]. See also [CR] for an explicit description of Λ when G is cyclic. But in the other cases almost nothing is known.

The Cohn localization is an interesting construction in algebra, but this theory plays an important role in homotopy theory, algebraic K-theory and homology surgery.

The main example of a geometric problem involving the Cohn localization is the problem of codimension 2 embeddings. If we want to classify the embeddings from a closed manifold M to a closed manifold W , we need to determine the complement X of this embedding. The homology of X with coefficients in $\mathbf{Z}[\pi_1(W)]$ can be determined by the geometry of the embedding. If the codimension of the embedding is more than 2, the fundamental group of X is isomorphic to $\pi_1(W)$ and the homotopy type of X can be determined (in some sense). But, in the codimension 2 case, the situation is completely different because the morphism $\pi_1(X) \rightarrow \pi_1(W)$ is never an isomorphism.

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There is a strategy to study the problem in this case (see [CS] and [LD]):

First of all, we have to choose a group π and a morphism $\pi \rightarrow \pi_1(W)$ which is possibly a morphism $\pi_1(X) \rightarrow \pi_1(W)$ obtained from an embedding. Then we set: $A = \mathbf{Z}[\pi]$, $B = \mathbf{Z}[\pi_1(W)]$. We have a ring homomorphism $A \rightarrow B$ and then a localization ring $\Lambda = L(A \rightarrow B)$.

Second step: for a possible space X and a possible map $X \rightarrow W$, we have to find a space Y between X and W which is universal with respect to the following property:

- the map $X \rightarrow Y$ induces isomorphisms $\pi_1(X) \xrightarrow{\sim} \pi_1(Y)$ and $H_*(X, B) \xrightarrow{\sim} H_*(Y, B)$.

Such a space Y is obtained by localization in the category of spaces over W and the homotopy groups of Y are Λ -modules.

Last step: find a manifold M' and a B -homology equivalence $M' \rightarrow Y$. The obstruction to finding such a manifold lies, roughly speaking, in a surgery group $\Gamma_n(A \rightarrow B)$ defined by Cappell and Shaneson in [CS] which is isomorphic to the classical surgery obstruction group $L_n(\Lambda)$ [V2].

If all this procedure works M' will be the complement of a tubular neighbourhood of an embedding $M \subset W$.

The Cohn localization ring Λ appears twice in this theory: in the homotopy groups of Y and in the obstruction surgery group.

The first result of this paper is a complete description of $L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ when G is a finite group.

Theorem A. *Let G be a finite group. For every prime p , denote by G_p the quotient of G by the subgroup of G generated by all the elements of order coprime to p , and by $\varepsilon_p : \mathbf{Z}_{(p)}[G_p] \rightarrow \mathbf{Z}_{(p)}$ the corresponding augmentation map. Then the Cohn localization ring $\Lambda = L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ is given by the following pull-back diagram:*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \prod_p \mathbf{Z}_{(p)}[G_p] \\ \downarrow & & \downarrow \Pi_p \varepsilon_p \\ \mathbf{Z} & \xrightarrow{\Delta} & \prod_p \mathbf{Z}_{(p)} \end{array}$$

where Δ is the diagonal inclusion and the product is over all non-trivial G_p .

Consider a ring homomorphism $A \rightarrow B$. We'll say that B is a central localization (resp. an Ore localization) of A if B is the ring $S^{-1}A$ where S is a multiplicative set in the center of A (resp. a multiplicative set in A satisfying the Ore condition).

We say that B is stably flat over A if the two conditions hold:

- the multiplication map: $B \otimes_A B \rightarrow B$ is an isomorphism
- $\text{Tor}_i^A(B, B) = 0$ for all $i > 0$.

The second result is the following:

Theorem B. *Let G be a finite group and Λ be the Cohn localization ring $L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$. Then the following conditions are equivalent:*

- 1) Λ is a central localization of $\mathbf{Z}[G]$
- 2) Λ is an Ore localization of $\mathbf{Z}[G]$
- 3) Λ is a flat left $\mathbf{Z}[G]$ -module
- 4) Λ is stably flat over $\mathbf{Z}[G]$
- 5) G is nilpotent.

Notice that 2) and 3) are equivalent in any case by a result of Teichner [T].

The section 1 is devoted to a brief presentation of the Cohn localization and some useful properties of this functor.

In section 2, the localization functor is extended to the category of (chain) complexes over a ring A . The universal property of this localization functor Φ and the ring structure of A induce a graded ring structure on the homology Λ_* of the localization complex $\Phi(A)$. The subring Λ_0 of the graded localization ring Λ_* is actually the Cohn localization of A . Moreover, for each complex C , there is a spectral sequence converging to $H_*(\Phi(C))$, where the E^2 term is given by: $E_{pq}^2 = H_p(C \otimes_A \Lambda_q)$.

The last section is devoted to the Cohn localization ring $\Lambda = L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ when G is any finite group. Using results of sections 1 and 2, Theorem A and Theorem B are proven and a complete description of the graded localization ring Λ_* is given (Theorem 3.8). There is also some examples and counter-examples at the end of the paper.

1. Cohn localization of rings.

For any ring A , the class of finitely generated projective right A -modules will be denoted by $\mathcal{P}(A)$. The set of matrices with entries in A will be denoted by $M(A)$ and the class of morphisms between two modules in $\mathcal{P}(A)$ will be denoted by $\mathcal{M}(A)$. The set $M(A)$ may be seen as a subset of $\mathcal{M}(A)$.

Let A be a ring and W be a class of morphisms in $\mathcal{M}(A)$. The ring obtained from A by formally inverting all morphisms in W is called the Cohn localization (or the universal localization) of A with respect to W . This localization ring Λ is equipped with a ring homomorphism from A to Λ and the morphism $A \rightarrow \Lambda$ is characterized by the following properties:

- every morphism in W is sent to an isomorphism in $\mathcal{M}(\Lambda)$.
- Λ is universal with respect to this property.

This localization ring will be denoted by $L(A, W)$ or $W^{-1}A$.

This universal characterization implies that the ring $L(A, W)$ is unique up to canonical isomorphism if it exists. A construction of $L(A, W)$ by generators and relations is given in [C1], [C2] if W is a set or matrices. Actually this construction may be modified in order to produce $L(A, W)$ in the general case. See also a presentation

of the Cohn localization in [R].

On the other hand, a construction of this ring will be obtained as a consequence of Theorem 2.7 in the next section.

Let $f : A \rightarrow B$ be a ring homomorphism. There is an exact sequence in algebraic K-theory:

$$\cdots \longrightarrow K_1(A) \longrightarrow K_1(B) \longrightarrow K_1(f) \xrightarrow{\partial} K_0(A) \longrightarrow K_0(B) \longrightarrow \cdots$$

and the relative K-theory group $K_1(f) = K_1(B, A)$ is generated by triples (P, Q, λ) , where P and Q are in $\mathcal{P}(A)$ and λ is an isomorphism from $P \otimes_A B$ to $Q \otimes_A B$.

If (P, Q, λ) is such a triple, the class of (P, Q, λ) in $K_1(f)$ will be denoted by $\tau(P, Q, \lambda)$.

Let u be a subgroup of $K_1(f)$. The class of morphisms $\alpha : P \rightarrow Q$ in $\mathcal{M}(A)$ such that $\alpha \otimes_A B$ is an isomorphism and $\tau(P, Q, \alpha \otimes_A B)$ belongs to u will be denoted by $W^u(A \rightarrow B)$ and the localization ring of A with respect to this class will be denoted by $L^u(A \rightarrow B)$.

If u is the trivial subgroup of $K_1(f)$ (resp. the all group $K_1(f)$, the image of $K_1(B) \rightarrow K_1(f)$), the localization ring $L^u(A \rightarrow B)$ will be denoted by $L^s(A \rightarrow B)$ (resp. $L^p(A \rightarrow B)$, $L^h(A \rightarrow B)$).

If $u \subset v$ are two subgroups of $K_1(f)$, we have: $W^u(A \rightarrow B) \subset W^v(A \rightarrow B)$ and there is a canonical ring homomorphism: $L^u(A \rightarrow B) \rightarrow L^v(A \rightarrow B)$.

If the ring $L^u(A \rightarrow B)$ doesn't depend of u it will be denoted by $L(A \rightarrow B)$.

Remark: Actually every Cohn localization ring $L(A, W)$ is the localization ring $L^u(A \rightarrow B)$ for some ring homomorphism $A \rightarrow B$ and some subgroup of $K_1(A \rightarrow B)$. More precisely we have the following:

Let A be a ring and W be a set of morphisms in $\mathcal{M}(A)$. Let Λ be the localization ring $L(A, W)$ and u be a subgroup of $K_1(A \rightarrow \Lambda)$ containing all the elements $\tau(P, Q, \alpha \otimes_A \Lambda)$ with $\alpha : P \rightarrow Q$ in W . Then Λ is the localization ring $L^u(A \rightarrow \Lambda)$.

1.1 Lemma: *Let $f : A \rightarrow B$ be a ring homomorphism. Let W^h be the set of square matrices in $M(A)$ becoming invertible in $M(B)$ and W^s be the set of square matrices $M \in M(A)$ such that $f(M)$ is an invertible matrix with zero torsion in $K_1(B)$. Then we have: $L^h(A \rightarrow B) = L(A, W^h)$ and $L^s(A \rightarrow B) = L(A, W^s)$.*

Proof: Denote by Λ^s and Λ^h the localization rings $L(A, W^s)$ and $L(A, W^h)$.

Let u be the trivial group in $K_1(f)$ and v be the image of $K_1(B) \rightarrow K_1(f)$. The sets W^h and W^s are included in $W^u(A \rightarrow B)$ and $W^v(A \rightarrow B)$. Then we have ring homomorphisms $\Lambda^s \rightarrow L^s(A \rightarrow B)$ and $\Lambda^h \rightarrow L^h(A \rightarrow B)$ and the only thing to do is to prove that every morphism in $W^u(A \rightarrow B)$ (resp. $W^v(A \rightarrow B)$) is sent to an isomorphism in $\mathcal{M}(\Lambda^s)$ (resp. in $\mathcal{M}(\Lambda^h)$).

Let $\alpha : P \rightarrow Q$ be a morphism in $W^v(A \rightarrow B)$ and θ be the class $\tau(P, Q, \alpha \otimes B)$ in $K_1(f)$. Then α belongs to $W^u(A \rightarrow B)$ if and only if $\theta = 0$. Since θ belongs to the image of $K_1(B) \rightarrow K_1(f)$, we have: $\partial(\theta) = 0$ in $K_0(A)$ and P and Q are equal in $K_0(A)$. Therefore there exists a module $K \in \mathcal{P}(A)$ such that $P \oplus K$ and

$Q \oplus K$ are isomorphic and $P \oplus K$ is free. Let β be the morphism $\alpha \oplus \text{Id}_K$. Then β is represented by a square matrix in W^h and β is invertible in $M(\Lambda^h)$. Hence α is sent to an isomorphism in $\mathcal{M}(\Lambda^h)$.

Suppose $\theta = 0$. Then α belongs to $W^u(A \rightarrow B)$ and β also. There exist a free module $F \in \mathcal{P}(A)$ and two isomorphisms $\varepsilon : F \xrightarrow{\sim} P \oplus K$ and $\varepsilon' : F \xrightarrow{\sim} Q \oplus K$ and we have:

$$\tau(P \oplus K, Q \oplus K, \beta \otimes B) = \tau(F, F, (\varepsilon'^{-1} \circ \beta \circ \varepsilon) \otimes B) = 0$$

Set: $\beta' = \varepsilon'^{-1} \circ \beta \circ \varepsilon$. Then the torsion τ' of $\beta' \otimes B$ in $K_1(B)$ vanishes in $K_1(f)$ and τ' comes from an element $\hat{\tau} \in K_1(A)$. Up to stabilize K and F we may as well suppose that $\hat{\tau}$ is the torsion of an automorphism $\hat{\varepsilon}$ in $\text{GL}(F)$. Consider the morphism: $\hat{\beta} = \hat{\varepsilon}^{-1} \circ \beta'$. This morphism corresponds to a square matrix with zero torsion and $\hat{\beta}$ is invertible in $M(\Lambda^s)$. Therefore β and α are sent to isomorphisms in $\mathcal{M}(\Lambda^s)$. \square

1.2 Examples: Let A be a commutative ring and W be a set of matrices in $M(A)$. Then we have the following:

If W contains some non-square matrix then the ring $L(A, W)$ is the trivial ring.

If W contains only square matrices the ring $L(A, W)$ is the localized ring $S^{-1}A$, where S is the multiplicative set in A generated by the determinants of the matrices in W .

Let $f : A \rightarrow B$ be a ring homomorphism between commutative rings. Consider the following subsets of A : the set $S = f^{-1}(1)$ and the set $\Sigma = f^{-1}(B^*)$, where B^* is the group of units in B . Then we have:

$$L^s(A \rightarrow B) = S^{-1}A$$

$$L^h(A \rightarrow B) = \Sigma^{-1}A$$

In the non commutative case, the localization ring is much more difficult to understand. Nevertheless it is possible to give some description of morphisms in $\mathcal{M}(\Lambda)$. In order to do that, we need some notations:

Notation: Let A be a ring and $\alpha, \beta_1, \beta_2, \dots, \beta_p$ be some morphisms in $\mathcal{M}(A)$. We say that α is a compatible product $\beta_1\beta_2 \dots \beta_n$ if there are finitely generated projective right A -modules P_0, P_1, \dots, P_n such that β_i is a morphism from P_i to P_{i-1} , α is a morphism from P_n to P_0 and α is equal to the composite $\beta_1\beta_2 \dots \beta_p$.

Notation: Let A be a ring and W be a class in $\mathcal{M}(A)$. The class of morphisms in $\mathcal{M}(A)$ of the form:

$$\begin{pmatrix} \alpha_1 & * & . & . & . & * \\ 0 & \alpha_2 & . & . & . & * \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & * \\ 0 & 0 & . & . & 0 & \alpha_p \end{pmatrix}$$

where each α_i is in W and each $*$ is in $\mathcal{M}(A)$, will be denoted by \widehat{W} .

The classes W and \widehat{W} induce the same localization ring.

1.3 Proposition: Let A be a ring, W be a class of morphisms in $\mathcal{M}(A)$ and Λ be the ring $L(A, W)$. Denote by M_0 the image of $\mathcal{M}(A)$ in $\mathcal{M}(\Lambda)$ and by M_1 the image of \widehat{W} in $\mathcal{M}(\Lambda)$. Every morphism in M_1 is clearly invertible.

Then for every P and Q in $\mathcal{P}(A)$, every morphism from $P \otimes_A \Lambda$ to $Q \otimes_A \Lambda$ is a compatible product $\lambda\alpha^{-1}\mu$ with λ and μ in M_0 and α in M_1 .

Proof: Let \mathcal{C} be the class of all Λ -modules on the form $M \otimes_A \Lambda$ with $M \in \mathcal{P}(A)$ and \mathcal{M} be the class of morphisms in $\mathcal{M}(\Lambda)$ between two modules in \mathcal{C} . Denote by $\mathcal{M}'(\Lambda)$ the class of compatible products $\lambda\alpha^{-1}\mu$ with λ, μ in M_0 and α in M_1 . Clearly, $\mathcal{M}'(\Lambda)$ is contained in \mathcal{M} . Because of the following:

$$\lambda\alpha^{-1}\mu + \lambda'\alpha'^{-1}\mu' = (\lambda \quad \lambda') \begin{pmatrix} \alpha & 0 \\ 0 & \alpha' \end{pmatrix}^{-1} \begin{pmatrix} \mu \\ \mu' \end{pmatrix}$$

the class $\mathcal{M}'(\Lambda)$ is stable under (compatible) sum. Moreover we have:

$$(\lambda\alpha^{-1}\mu)(\lambda'\alpha'^{-1}\mu') = (-\lambda \quad 0) \begin{pmatrix} \alpha & \mu\lambda' \\ 0 & \alpha' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \mu' \end{pmatrix}$$

and $\mathcal{M}'(\Lambda)$ is stable under compatible product.

In particular the set of 1×1 -matrices in $\mathcal{M}'(\Lambda)$ is a subring Λ_0 of Λ containing the image of $A \rightarrow \Lambda$. Therefore every matrix in $\mathcal{M}'(\Lambda)$ is a matrix in $\mathcal{M}(\Lambda_0)$ and $\mathcal{M}'(\Lambda)$ is contained in $\mathcal{M}(\Lambda_0)$.

On the other hand every morphism in the image of $W \rightarrow \mathcal{M}'(\Lambda)$ is invertible in $\mathcal{M}'(\Lambda) \subset \mathcal{M}(\Lambda_0)$ and the ring Λ_0 satisfies the universal property of Λ . Hence one has: $\Lambda = \Lambda_0$ and the class $\mathcal{M}'(\Lambda)$ is the class of morphisms in $\mathcal{M}(\Lambda)$ between two Λ -modules in \mathcal{C} . The result follows. \square

1.4 Corollary: Let $A \rightarrow B$ be a ring homomorphism and u be a subgroup of $K_1(A \rightarrow B)$. Let Λ be the Cohn localization ring $L^u(A \rightarrow B)$. Then for every module $P \in \mathcal{P}(\Lambda)$ and every morphism $\varphi : P \rightarrow P$, the following holds:

$$\varphi \otimes_{\Lambda} B = Id_{P \otimes B} \implies \varphi \text{ is an isomorphism.}$$

Proof: Let W be the class $W^u(A \rightarrow B)$. The class \widehat{W} is actually the class W .

Let $\varphi : P \rightarrow P$ be a morphism in $\mathcal{M}(\Lambda)$ such that $\varphi \otimes_{\Lambda} B$ is the identity. We have to prove that φ is an isomorphism.

First of all, suppose P is free. Then there is a free module $F \in \mathcal{P}(A)$ such that: $P = F \otimes_A \Lambda$ and $\varphi \otimes_{\Lambda} B$ is the identity of $F \otimes_A B$.

Because of Proposition 1.3, there exist two modules P_1 and Q_1 in $\mathcal{P}(A)$ and three morphisms:

$$F \xrightarrow{\lambda} P_1 \xleftarrow{\alpha} Q_1 \xrightarrow{\mu} F$$

such that α belongs to W and we have: $\varphi = \lambda \otimes_A \Lambda \circ (\alpha \otimes_A \Lambda)^{-1} \circ \mu \otimes_A \Lambda$.

Denote by λ', α', μ' (resp. $\lambda'', \alpha'', \mu''$) the images of λ, α, μ by the tensorization functor $\otimes_A \Lambda$ (resp. $\otimes_A B$). Then we have: $\varphi = \lambda' \alpha'^{-1} \mu'$.

Let ψ be the morphism $\begin{pmatrix} 0 & \lambda \\ \mu & \alpha \end{pmatrix}$ from $F \oplus Q_1$ to $F \oplus P_1$. By tensorization with Λ and B , we get the morphisms ψ' and ψ'' .

Since $\varphi \otimes_A B = \lambda'' \alpha''^{-1} \mu''$ is the identity, the morphism ψ'' is invertible with inverse:

$$\psi''^{-1} = \begin{pmatrix} 0 & \lambda'' \\ \mu'' & \alpha'' \end{pmatrix}^{-1} = \begin{pmatrix} -1 & \lambda'' \alpha''^{-1} \\ \alpha''^{-1} \mu'' & (1 - \alpha''^{-1} \mu'' \lambda'') \alpha''^{-1} \end{pmatrix}$$

An elementary computation shows the following in $K_1(A \rightarrow B)$:

$$\tau(F \oplus Q_1, F \oplus P_1, \psi'') = \tau(Q_1, P_1, \alpha'')$$

and $\tau(F \oplus Q_1, F \oplus P_1, \psi')$ belongs to u . By the universal property of Λ , ψ' is invertible.

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the inverse of ψ' . We have the following relations:

$$\begin{aligned} \lambda' c &= 1 & \lambda' d &= 0 & \mu' a + \alpha' c &= 0 & \mu' b + \alpha' d &= 1 \\ b \mu' &= 1 & a \lambda' + b \alpha' &= 0 & d \mu' &= 0 & c \lambda' + d \alpha' &= 1 \end{aligned}$$

So we have:

$$\begin{aligned} 0 &= \lambda' \alpha'^{-1} (\mu' a + \alpha' c) = \varphi a + \lambda' c = \varphi a + 1 \\ 0 &= (a \lambda' + b \alpha') \alpha'^{-1} \mu' = a \varphi + b \mu' = a \varphi + 1 \end{aligned}$$

Therefore φ is invertible (with inverse $-a$).

Consider now the general case.

Let Q be a module in $\mathcal{P}(\Lambda)$ such that $P \oplus Q$ is free. Because of the previous study, $\varphi \oplus \text{Id}_Q$ is invertible and φ is invertible too. The result follows. \square

1.5 Proposition: *Let $A \rightarrow B$ be a ring homomorphism and u be a subgroup of $K_1(A \rightarrow B)$. Let $\Lambda = L^u(A \rightarrow B)$ be the corresponding Cohn localization ring. Suppose the induced morphism $\Lambda \rightarrow B$ is surjective. Then, for every finitely generated right Λ -module M , the following holds:*

$$M \otimes_\Lambda B = 0 \implies M = 0$$

Proof: Suppose M is a finitely generated right Λ -module and $M \otimes_\Lambda B$ is trivial. Take a free resolution of M :

$$C_1 \xrightarrow{d} C_0 \longrightarrow M \longrightarrow 0$$

where C_0 is finitely generated. Since $M \otimes_\Lambda B = 0$ the morphism

$$C_1 \otimes_\Lambda B \xrightarrow{d} C_0 \otimes_\Lambda B$$

is surjective and has a section s . Since $\Lambda \rightarrow B$ is surjective s can be lifted to a morphism $\tilde{s} : C_0 \rightarrow C_1$. The composite morphism $d \circ \tilde{s}$ corresponds to a matrix in $M(\Lambda)$ which is sent to an identity matrix in $M(B)$. Because of Corollary 1.4, u is invertible and the map $d : C_1 \rightarrow C_0$ is surjective. The result follows. \square

1.6 Corollary: *Let $A \rightarrow B$ be a ring homomorphism and u be a subgroup of $K_1(A \rightarrow B)$.*

Suppose the morphism $L^u(A \rightarrow B) \rightarrow B$ is onto. Then, for every subgroup v of $K_1(A \rightarrow B)$ containing u the canonical ring homomorphism from $L^u(A \rightarrow B)$ to $L^v(A \rightarrow B)$ is an isomorphism.

Proof: Suppose $L^u(A \rightarrow B) \rightarrow B$ is onto. Let $\varphi : P \rightarrow Q$ be a morphism in $\mathcal{M}(\Lambda)$ such that $\varphi \otimes_\Lambda B$ is an isomorphism. Let K and C be the kernel and the cokernel of φ . The module C is finitely generated and $C \otimes_\Lambda B$ is trivial. Because of Proposition 1.5, C is trivial too and φ is surjective. Hence the sequence:

$$0 \longrightarrow K \longrightarrow P \longrightarrow Q \longrightarrow 0$$

is split exact and K is finitely generated. Then we have a split exact sequence:

$$0 \longrightarrow K \otimes_\Lambda B \longrightarrow P \otimes_\Lambda B \xrightarrow{\sim} Q \otimes_\Lambda B \longrightarrow 0$$

and the module $K \otimes_\Lambda B$ is trivial. Because of Proposition 1.5 the module K is also trivial and φ is an isomorphism.

Therefore $L^u(A \rightarrow B)$ satisfies the universal property of $L^v(A \rightarrow B)$ and the result follows. \square

1.7 Remark: If the morphism $A \rightarrow B$ is onto, the ring $L^u(A \rightarrow B)$ doesn't depend on u and will be denoted by $L(A \rightarrow B)$.

1.8 Corollary: *Let G be a group and H be a finitely generated perfect subgroup of G . Let N be the normalizer of H . Then the morphism $\mathbf{Z}[G] \rightarrow L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ factors through $\mathbf{Z}[G/N]$ and $L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ is the ring $L(\mathbf{Z}[G/N] \rightarrow \mathbf{Z})$.*

Proof: Denote by Λ the localization ring $L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ and by $f : \mathbf{Z}[G] \rightarrow \Lambda$ the induced morphism. Let I be the kernel of the morphism $\mathbf{Z}[G] \rightarrow \mathbf{Z}[H \setminus G]$. Since H is finitely generated, I is a finitely generated right $\mathbf{Z}[G]$ -module and $J = I \otimes_{\mathbf{Z}[G]} \Lambda$ is a finitely generated right Λ -module. We have an exact sequence:

$$0 \longrightarrow I \longrightarrow \mathbf{Z}[G] \longrightarrow \mathbf{Z}[H \setminus G] \rightarrow 0$$

and that implies:

$$\begin{aligned} I \otimes_{\mathbf{Z}[G]} \mathbf{Z} &\simeq \text{Tor}_1^{\mathbf{Z}[G]}(\mathbf{Z}[H \setminus G], \mathbf{Z}) \simeq H_1(H) \simeq 0 \\ \implies J \otimes_\Lambda \mathbf{Z} &= I \otimes_{\mathbf{Z}[G]} \Lambda \otimes_\Lambda \mathbf{Z} = I \otimes_{\mathbf{Z}[G]} \mathbf{Z} = 0 \end{aligned}$$

Hence: $J = 0$ because of Proposition 1.5.

On the other hand, the tensorization by Λ induces a commutative diagram:

$$\begin{array}{ccc} I & \longrightarrow & \mathbf{Z}[G] \\ \downarrow & & \downarrow \\ J & \longrightarrow & \Lambda \end{array}$$

Since J is trivial, I vanishes in Λ . Thus, for every h in H , $h - 1$ vanishes in Λ and we have: $f(H) = \{1\}$ and therefore: $f(N) = \{1\}$. Hence the morphism f factors through $\mathbf{Z}[G/N]$. Moreover the morphism $\mathbf{Z}[G/N] \rightarrow \Lambda$ satisfies the universal condition defining the ring $L(\mathbf{Z}[G/N] \rightarrow \mathbf{Z})$ and Λ is the localization ring $L(\mathbf{Z}[G/N] \rightarrow \mathbf{Z})$. \square

1.9 Lemma: *Let $f : A \rightarrow B$ be a surjective ring homomorphism. Then the map $A \rightarrow L(A \rightarrow B)$ is an isomorphism if and only if every element in $f^{-1}(1)$ is invertible.*

Proof: Denote by Λ the localization ring $L(A \rightarrow B)$. Suppose $A \rightarrow \Lambda$ is an isomorphism. Let a be an element in $f^{-1}(1)$. This element represents a 1×1 -matrix which is invertible in $M(B)$ and then in $M(\Lambda)$. Because $A \rightarrow \Lambda$ is bijective, the matrix is invertible. Thus a is invertible and $f^{-1}(1)$ is contained in A^* (the group of units of A).

Suppose now $f^{-1}(1)$ is contained in A^* . Consider a square matrix M in $M(A)$ such that $f(M)$ is invertible. Let N be a matrix in $M(A)$ with $f(N) = f(M)^{-1}$. Since the diagonal entries of MN are invertible in A and the other entries are killed in B , it is possible to multiply MN on the left and the right by elementary matrices in order to obtain a diagonal matrix and the same holds for NM . Therefore MN and NM are invertible and M is invertible too. Then every square matrix in $M(A)$ sent to an invertible matrix in $M(B)$ is invertible and, because of Lemma 1.1, we have:

$$L(A \rightarrow B) = L^h(A \rightarrow B) = A \quad \square$$

As a corollary one gets the following result:

1.10 Proposition: *Let p be a prime and G be a finite p -group. Denote by Λ the localization ring $L(\mathbf{Z}_{(p)}[G] \rightarrow \mathbf{Z}_{(p)})$. Then the map $\mathbf{Z}_{(p)}[G] \rightarrow \Lambda$ is an isomorphism.*

Proof: Let's say that a ring homomorphism f is local if f is surjective and every element in $f^{-1}(1)$ is invertible.

The proposition is obvious if the order of G is 1. The result will be proven by induction. Suppose the order of G is $p^n > 1$ and the proposition is true for every group of order p^i , $i < n$. Let $z \in G$ be a central element of order p and G' be the quotient $G / \langle z \rangle$. Consider the maps:

$$\mathbf{Z}_{(p)}[G] \xrightarrow{f} \mathbf{Z}_{(p)}[G'] \xrightarrow{g} \mathbf{Z}_{(p)}$$

Because of lemma 1.9 g is local and we have to prove that $g \circ f$ is also local.

Let f' be the reduction of $f \bmod p$:

$$f' : \mathbf{F}_p[G] \longrightarrow \mathbf{F}_p[G']$$

Set: $z = 1 + u$. The relation: $z^p = 1$ becomes mod p : $u^p = 0$. Let U be an element in $f'^{-1}(1)$. This element has the form: $U = 1 - uV$ for some element $V \in \mathbf{F}_p[G]$. Then U is invertible with inverse:

$$U^{-1} = 1 + uV + u^2V^2 + \dots + u^{p-1}V^{p-1}$$

and f' is local. Let U be an element of $f^{-1}(1) \subset \mathbf{Z}_{(p)}[G]$. The multiplication by U is an endomorphism φ on $\mathbf{Z}_{(p)}[G]$ considered as a free $\mathbf{Z}_{(p)}$ -module. Because U is invertible mod p , the determinant of φ is non zero mod p and then invertible in $\mathbf{Z}_{(p)}$. Consequently, U is invertible and f and $g \circ f$ are local. The result follows. \square

We have a last result in this section which will be used in the sequel.

1.11 Proposition: *Let $f : A \rightarrow B$ and $g : B \rightarrow R$ be two ring homomorphisms. Suppose f is onto. Let I be the kernel of f and Λ be the Cohn localization $L^h(A \rightarrow R)$. Let J be the two-sided ideal of Λ generated by the image of I in Λ . Then the Cohn localization $L^h(B \rightarrow R)$ is the quotient Λ/J .*

Proof: Since $I \subset A$ is killed in R , J is also killed in R and we have a commutative diagram of rings:

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & R \\ \downarrow & & \downarrow & & \downarrow \text{Id} \\ \Lambda & \longrightarrow & \Lambda/J & \longrightarrow & R \end{array}$$

Let M be a square matrix in $M(B)$ which is invertible in $M(R)$. Since f is onto, M can be lifted in a matrix $N \in M(A)$ which is invertible in $M(\Lambda)$ and then in $M(\Lambda/J)$. Therefore M is invertible in $M(\Lambda/J)$.

Let $h : B \rightarrow C$ be a ring homomorphism such that every square matrix in $M(B)$ becoming invertible in $M(R)$ is invertible in $M(C)$. Be the universal property, the morphism $h \circ f : A \rightarrow C$ factors through Λ . Moreover J is killed by the induced morphism $\Lambda \rightarrow C$ and this morphism factors through Λ/J . So we have the following

commutative diagram of rings:

$$\begin{array}{ccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
\downarrow & & \downarrow & & \downarrow \text{Id} \\
\Lambda & \longrightarrow & \Lambda/J & \longrightarrow & C
\end{array}$$

and the morphism $B \rightarrow C$ factors in a unique way through Λ/J . By the universal property, Λ/J is the Cohn localization $L^h(B \rightarrow R)$. \square

2. Localization of complexes.

This section is devoted to the localization in the category of complexes. For similar theories see also [V2] section 5, [D], [R], [NR].

Let R be a ring. The graded differential right R -modules which are projective and bounded from below define a category $\mathcal{C}_*(R)$. The objects in $\mathcal{C}_*(R)$ are called complexes (or R -complexes if needed). A morphism in $\mathcal{C}_*(R)$ is a linear map of some degree which commutes with the differentials (in the graded sense). A morphism in $\mathcal{C}_*(R)$ is a cofibration if it is injective with cokernel in $\mathcal{C}_*(R)$. A complex C is said to be finite if it is bounded, finitely generated and free in each degree. A complex C is said to be finitely dominated (or compact) if C is bounded and finitely generated.

A complex C is finitely dominated if and only if it is a direct summand of a finite complex.

Let C and C' be two R -complexes. Denote by $\text{Hom}(C, C')_p$ the module of R -linear maps of degree p from C to C' . The direct sum of all these modules is a $(\mathbf{Z}-)$ graded module denoted by $\text{Hom}(C, C')$. There is a unique differential d on this graded module such that:

$$\forall (x, f) \in C \times \text{Hom}(C, C'), \quad d(f(x)) = d(f)(x) + (-1)^{|f|} f(d(x))$$

where $|f|$ is the degree of f . Equipped with this differential, $\text{Hom}(C, C')$ is a graded differential module. A cycle in $\text{Hom}(C, C')$ is a morphism and a boundary is a homotopy. Moreover if C, C' and C'' are R -complexes, the composition map:

$$\text{Hom}(C', C'') \otimes \text{Hom}(C, C') \longrightarrow \text{Hom}(C, C'')$$

is a degree 0 morphism of graded differential modules.

2.0 Lemma: *Let C and C' be two R -complexes. Then one has the following properties:*

- *the graded differential module $\text{Hom}(C, C')$ is acyclic if and only if every morphism from C to C' is null-homotopic.*

- the graded differential module $\text{Hom}(C, C)$ is a graded differential ring and $\text{Hom}(C, C')$ is a graded differential right $\text{Hom}(C, C)$ -module.

Proof: It's just a straightforward computation. \square

Throughout this section A is a ring and W is a class of morphisms in $\mathcal{M}(A)$. The Cohn localization $L(A, W)$ will be denoted by Λ .

The class of complexes of length 2:

$$\dots \longrightarrow 0 \longrightarrow P \xrightarrow{d} Q \longrightarrow 0 \longrightarrow \dots$$

such that the only non zero differential d lies in W will be denoted by \mathcal{W}_0 .

Definitions: A right A -module M is said to be W -local if, for every morphism $f : P \rightarrow Q$ in W , the induced map $f^* : \text{Hom}(Q, M) \rightarrow \text{Hom}(P, M)$ is bijective.

An A -complex C is said to be W -local if every morphism from a complex in \mathcal{W}_0 to C is null-homotopic.

Let \mathcal{C} be a class of A -complexes. An A -complex C is said to be \mathcal{C} -acyclic if every morphism from C to a complex in \mathcal{C} is null-homotopic.

The two notions of locality are related by the following result:

2.1 Lemma: Let C be an A -complex. Then C is W -local if and only if its homology is W -local.

Proof: For simplicity the module $H_i(C)$ will be denoted by H_i .

Let K be a complex in \mathcal{W}_0 concentrated in degrees p and $p+1$. Denote by $f : K_{p+1} \rightarrow K_p$ the non zero differential of K and set:

$$C^K = \text{Hom}(K, C)$$

We have a short exact sequence of complexes:

$$0 \longrightarrow K_p \longrightarrow K \longrightarrow K_{p+1} \longrightarrow 0$$

and then a short exact sequence of graded differential modules:

$$0 \longleftarrow \text{Hom}(K_p, C) \longleftarrow C^K \longleftarrow \text{Hom}(K_{p+1}, C) \longleftarrow 0$$

inducing a long exact sequence:

$$\dots \rightarrow H_i(C^K) \rightarrow \text{Hom}(K_p, H_{p+i}) \xrightarrow{f^*} \text{Hom}(K_{p+1}, H_{p+i}) \rightarrow H_{i-1}(C^K) \rightarrow \dots$$

The graded module $C^K = \text{Hom}(K, C)$ has trivial homology if and only if every morphism from K to C is null-homotopic. The desired result follows. \square

Notations: The class of W -local complexes in $\mathcal{C}_*(A)$ will be denoted by \mathcal{L} . The class of \mathcal{L} -acyclic complexes in $\mathcal{C}_*(A)$ will be denoted by \mathcal{W} . A morphism between two A -complexes will be called a \mathcal{W} -equivalence if its mapping cone belongs to \mathcal{W} .

2.2 Lemma: *The class \mathcal{W} has the following properties:*

- \mathcal{W} contains \mathcal{W}_0 and the acyclic complexes.
- if $0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$ is a short exact sequence of complexes, then if two of these complexes are in \mathcal{W} the third one is in \mathcal{W} too.
- If a complex C is a direct sum of complexes in \mathcal{W} , then C belongs to \mathcal{W} .
- If a complex C is the colimit of an infinite sequence:

$$C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow \cdots$$

such that all these maps are cofibrations and all the complexes C_i are in \mathcal{W} , then C belongs to \mathcal{W} .

Proof: The first property is obvious.

Consider a short exact sequence in $\mathcal{C}_*(A)$:

$$0 \rightarrow C \rightarrow C' \rightarrow C'' \rightarrow 0$$

Let L be a W -local complex. Then we have a short exact sequence of graded differential modules:

$$0 \leftarrow \text{Hom}(C, L) \leftarrow \text{Hom}(C', L) \leftarrow \text{Hom}(C'', L) \leftarrow 0$$

If two of the complexes are in \mathcal{W} , two of the graded differential modules $\text{Hom}(C, L)$, $\text{Hom}(C', L)$, $\text{Hom}(C'', L)$ are acyclic. Then the third one is also acyclic and the third complex lies in \mathcal{W} .

Suppose C is a complex on the form:

$$C = \bigoplus_i C_i$$

where each C_i lies in \mathcal{W} . Since C is a complex, there is an integer p such that each C_i vanishes in degree less than p . Let L be a W -local complex. We have:

$$\text{Hom}(C, L) = \prod_i \text{Hom}(C_i, L)$$

Since C_i lies in \mathcal{W} , $\text{Hom}(C_i, L)$ has trivial homology. Therefore the homology of $\text{Hom}(C, L)$ is also trivial and every morphism from C to L is null-homotopic. Hence C belongs to \mathcal{W} .

Suppose C is the colimit of an infinite sequence:

$$C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow \cdots$$

such that all these maps are cofibrations and each C_i is in \mathcal{W} . Then there is an integer p such that C and each C_i vanishes in degree less than p . Let L be a W -local complex. We have an isomorphism of graded differential modules:

$$\text{Hom}(C, L) \simeq \varprojlim_k \text{Hom}(C_k, L)$$

and then an exact sequence:

$$0 \longrightarrow \varprojlim_k^1 H_{i+1}(\text{Hom}(C_k, L)) \longrightarrow H_i(\text{Hom}(C, L)) \longrightarrow \varprojlim_k H_i(\text{Hom}(C_k, L)) \longrightarrow 0$$

Since each C_k lies in \mathcal{W} , each $\text{Hom}(C_k, L)$ is acyclic and $\text{Hom}(C, L)$ also. Therefore C lies in \mathcal{W} . \square

2.3 Lemma: *The class \mathcal{L} has the following properties:*

- \mathcal{L} contains the acyclic complexes.
- If $0 \longrightarrow C \longrightarrow C' \longrightarrow C'' \longrightarrow 0$ is a short exact sequence of complexes, then if two of these complexes are in \mathcal{L} , the third one is in \mathcal{L} too.
- If a complex C is a direct sum of complexes in \mathcal{L} , then C belongs to \mathcal{L} .
- Suppose a complex C is the colimit of an infinite sequence:

$$C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow \dots$$

such that all these maps are cofibrations and each C_i is in \mathcal{L} . Then C belongs to \mathcal{L} .

Proof: The first property is obvious.

Consider a short exact sequence of complexes:

$$0 \longrightarrow C \longrightarrow C' \longrightarrow C'' \longrightarrow 0$$

If K is a complex in \mathcal{W}_0 , we have a short exact sequence:

$$0 \longrightarrow \text{Hom}(K, C) \longrightarrow \text{Hom}(K, C') \longrightarrow \text{Hom}(K, C'') \longrightarrow 0$$

If two of the complexes C, C', C'' are in \mathcal{L} , two of the graded differential modules $\text{Hom}(K, C), \text{Hom}(K, C'), \text{Hom}(K, C'')$ are acyclic and the third one is also acyclic. Thus the third complex is in \mathcal{L} .

Suppose C is the direct sum of complexes C_i in \mathcal{L} . Let K be a complex in \mathcal{W}_0 . Since K is finitely dominated, $\text{Hom}(K, C)$ is the direct sum of the graded differential modules $\text{Hom}(K, C_i)$. Therefore $\text{Hom}(K, C)$ is acyclic and C belongs to \mathcal{L} .

Suppose C is the colimit of the sequence: $C_1 \rightarrow C_2 \rightarrow \dots$. Let K be a complex in \mathcal{W}_0 . Since K is finitely dominated, $\text{Hom}(K, C)$ is the colimit of the $\text{Hom}(K, C_i)$ and $H_*(\text{Hom}(K, C))$ is the colimit of the $H_*(\text{Hom}(K, C_i))$. Therefore $\text{Hom}(K, C)$ is acyclic and C belongs to \mathcal{L} . \square

Definition: Let $C \rightarrow C'$ be a morphism in $\mathcal{C}_*(A)$. We say that C' (or more precisely $C \rightarrow C'$) is a W -localization of C if the following holds:

- the morphism $C \rightarrow C'$ is a cofibration.
- C' belongs to \mathcal{L} .
- $C \rightarrow C'$ is a \mathcal{W} -equivalence.

2.4 Proposition: Let C be an A -complex and $\lambda_1 : C \rightarrow C_1$ and $\lambda_2 : C \rightarrow C_2$ be two W -localizations of C . Then there exists a homotopy equivalence $f : C_1 \rightarrow C_2$,

unique up to homotopy, making the following diagram commutative:

$$\begin{array}{ccc} & & C_1 \\ & \nearrow \lambda_1 & \downarrow f \\ C & & \\ & \searrow \lambda_2 & \downarrow \\ & & C_2 \end{array}$$

Proof: Let K_1 be the Cokernel of λ_1 . We have the following exact sequences:

$$0 \longrightarrow C \longrightarrow C_1 \longrightarrow K_1 \longrightarrow 0$$

$$0 \longleftarrow \text{Hom}(C, C_2) \longleftarrow \text{Hom}(C_1, C_2) \longleftarrow \text{Hom}(K_1, C_2) \longleftarrow 0$$

Since C_2 is in \mathcal{L} and K_1 in \mathcal{W} , the homology of $\text{Hom}(K_1, C_2)$ is trivial and the restriction map:

$$\text{Hom}(C_1, C_2) \longrightarrow \text{Hom}(C, C_2)$$

is an isomorphism in homology. Therefore the morphism λ_2 can be lifted in a morphism $f : C_1 \rightarrow C_2$. Moreover this lifting is unique up to homotopy. Similarly, we produce a morphism $g : C_2 \rightarrow C_1$. By unicity up to homotopy of $f \circ g$ and $g \circ f$, these two maps have to be homotopic to identities. The result follows. \square

2.5 Theorem: *There exist a functor Φ from $\mathcal{C}_*(A)$ to itself and a morphism λ from the identity of $\mathcal{C}_*(A)$ to Φ with the following properties:*

- *For every complex C , $\lambda_C : C \rightarrow \Phi(C)$ is a W -localization of C .*
- *The functor Φ sends cofibration to cofibration.*
- *If a complex C vanishes in degree less than p , the same holds for $\Phi(C)$.*

Proof: Let's take a set X in $\mathcal{P}(A)$ such that, for every morphism $\alpha : P \rightarrow Q$ in W , P and Q are isomorphic to some modules in X . Denote by W' the set of morphisms $\alpha : P \rightarrow Q$ such that P and Q are in X and there exist two isomorphisms ε and ε' such that $\varepsilon \circ \alpha \circ \varepsilon'$ belongs to W .

The class of complexes of length 2 such that the only non zero differential is in W (resp. W') will be denoted by \mathcal{W}_0 (resp. \mathcal{W}'_0). Notice that the class \mathcal{W}'_0 is a set and every complex in \mathcal{W}_0 is isomorphic to a complex in \mathcal{W}'_0 .

Let C be a complex. Denote by $E(C)$ the set of pairs (K, φ) where K is a complex in \mathcal{W}'_0 and φ is a non zero morphism of degree 0 from K to C . The mapping cone of the map:

$$\oplus K \xrightarrow{\oplus \varphi} C$$

where the direct sum is over all $(K, \varphi) \in E(C)$, will be denoted by $\Phi_1(C)$ and the complex $\oplus K$ will be denoted by $\Psi(C)$. By construction we have an exact sequence of complexes:

$$0 \longrightarrow C \xrightarrow{i} \Phi_1(C) \xrightarrow{j} \Psi(C) \longrightarrow 0$$

where i and j are two morphisms of degree 0 and -1 respectively.

Suppose C vanishes in degree less than p . Then for every $(K, \varphi) \in E(C)$, φ is not zero and the complex K vanishes in degree less than $p - 1$. Thus $\Psi(C)$ vanishes in degree less than $p - 1$ and $\Phi_1(C)$ vanishes in degree less than p .

Let's denote by $[K, \varphi] \otimes K$ the component of K in $\Psi(C)$ corresponding to $(K, \varphi) \in E(C)$.

If $f : C \rightarrow C'$ is a morphism, we get a morphism f_* from $\Psi(C)$ to $\Psi(C')$ defined by:

$$\forall (K, \varphi) \in E(C), \forall x \in K, \quad f_*([K, \varphi] \otimes x) = \begin{cases} [K, f \circ \varphi] \otimes x & \text{if } f \circ \varphi \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Then Ψ is a functor sending cofibration to cofibration. Moreover the map sending $[K, \varphi] \otimes x$ to $\varphi(x)$ is a morphism μ from Ψ to the identity of $\mathcal{C}_*(A)$ and Φ_1 is the mapping cone of μ . Thus Φ_1 is a functor sending cofibration to cofibration.

The inclusion $C \subset \Phi_1(C)$ is a cofibration of degree 0 and every morphism from a complex in \mathcal{W}'_0 to C is null-homotopic in $\Phi_1(C)$. But every complex in \mathcal{W}_0 is isomorphic to some complex in \mathcal{W}'_0 . Hence every morphism from a complex in \mathcal{W}_0 to C is null-homotopic in $\Phi_1(C)$.

By iterating this construction, we get an infinite sequence:

$$C \rightarrow \Phi_1(C) \rightarrow \Phi_2(C) \rightarrow \Phi_3(C) \rightarrow \dots$$

where all these maps are cofibrations of degree 0. Moreover each $\Phi_n(C)$ vanishes in degree less than p and the colimit $\Phi(C)$ of this sequence is a well defined complex in $\mathcal{C}_*(A)$ vanishing in degree less than p .

It is clear that $\Phi_1(C)/C$ is isomorphic to a direct sum of complexes in \mathcal{W}_0 . Because of lemma 2.2, each $\Phi_{n+1}(C)/\Phi_n(C)$, each $\Phi_n(C)/C$ and $\Phi(C)/C$ belongs to \mathcal{W} .

Let K be a complex in \mathcal{W}_0 and f be a morphism from K to $\Phi(C)$. Since K is finitely dominated $f(K)$ is contained in some $\Phi_n(C)$ and the map $K \rightarrow \Phi_n(C)$ is null-homotopic in $\Phi_{n+1}(C)$. So every morphism from K to $\Phi(C)$ is null-homotopic. Therefore $\Phi(C)$ is W -local. \square

2.6 Proposition: *The functor Φ is homotopically exact in the following sense:*

- *If a complex C is the direct sum of complexes C_i , the morphism $\oplus_i \Phi(C_i) \rightarrow \Phi(C)$ is a homotopy equivalence.*

- *Let*

$$0 \longrightarrow C \xrightarrow{i} C' \xrightarrow{j} C'' \longrightarrow 0$$

be a short exact sequence of complexes. Then the morphism induced by j from $\Phi(C')/i_(\Phi(C))$ to $\Phi(C'')$ is a homotopy equivalence.*

Proof: Suppose C is the direct sum of complexes C_i . For each i we have an exact sequence:

$$0 \longrightarrow C_i \longrightarrow \Phi(C_i) \longrightarrow E_i \longrightarrow 0$$

where E_i lies in \mathcal{W} . So we get the following exact sequence:

$$0 \longrightarrow C \longrightarrow \oplus_i \Phi(C_i) \longrightarrow \oplus_i E_i \longrightarrow 0$$

By lemmas 2.2 and 2.3, $\oplus_i E_i$ belongs to \mathcal{W} and $\oplus_i \Phi(C_i)$ belongs to \mathcal{L} . Therefore $\oplus_i \Phi(C_i)$ is a W -localization of C and the map $\oplus \Phi(C_i) \rightarrow \Phi(C)$ is a homotopy equivalence.

Consider a short exact sequence:

$$0 \longrightarrow C \xrightarrow{i} C' \xrightarrow{j} C'' \longrightarrow 0$$

The map i induces a cofibration i_* from $\Phi(C)$ to $\Phi(C')$. Denote by Σ the quotient $\Phi(C')/i_*(\Phi(C))$. So we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{i} & C' & \xrightarrow{j} & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \lambda' & & \downarrow \lambda'' & & \\ 0 & \longrightarrow & \Phi(C) & \longrightarrow & \Phi(C') & \longrightarrow & \Sigma & \longrightarrow & 0 \end{array}$$

where the two lines are exact. Take a cofibration μ from C'' to a contractible complex U . So we get the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C & \xrightarrow{i} & C' & \xrightarrow{j} & C'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \lambda' \oplus \mu \circ j & & \downarrow \lambda'' \oplus \mu & & \\ 0 & \longrightarrow & \Phi(C) & \longrightarrow & \Phi(C') \oplus U & \longrightarrow & \Sigma \oplus U & \longrightarrow & 0 \end{array}$$

In this diagram, the two lines are exact and the vertical lines are cofibrations. Because of lemmas 2.2 and 2.3, $\Sigma \oplus U$ is local and the Cokernel of $\lambda'' \oplus \mu$ lies in \mathcal{W} . Therefore $C'' \rightarrow \Sigma \oplus U$ is a W -localization of C'' and $\Sigma \rightarrow \Phi(C'')$ is a homotopy equivalence. \square

2.7 Theorem: *There is a well defined graded ring Λ_* and a ring homomorphism $A \rightarrow \Lambda_0$ such that:*

- $\Lambda_i = 0$ for $i < 0$ and $A \rightarrow \Lambda_0$ is the Cohn localization of A with respect to W .
- if C is a W -localization of A (considered as a complex concentrated in degree 0), Λ_* is isomorphic to $H_*(C)$ as a graded right A -module.
- for every W -local complex C , the graded module $H_*(C)$ is equipped with a structure of right Λ_* -module and every morphism $f : C \rightarrow C'$ between W -local complexes induces a Λ_* -linear map from $H_*(C)$ to $H_*(C')$.
- for every complex C , there is a canonical spectral sequence with the following E^1 and E^2 terms:

$$E_{pq}^1 = C_p \otimes_A \Lambda_q \quad E_{pq}^2 = H_p(C \otimes_A \Lambda_q)$$

converging to $H_{p+q}(\Phi(C))$. Moreover this spectral sequence is natural in C and compatible with the Λ_* -action.

The graded ring Λ_* is well defined up to canonical isomorphism and will be called the graded localization of A with respect to W .

Proof: Let C be a W -localization of A (considered as a complex concentrated in degree 0). Denote by E the Cokernel of the cofibration $A \rightarrow C$. We have two exact sequences:

$$\begin{aligned} 0 \longrightarrow A \longrightarrow C \longrightarrow E \longrightarrow 0 \\ 0 \longleftarrow \text{Hom}(A, C) \longleftarrow \text{Hom}(C, C) \longleftarrow \text{Hom}(E, C) \longleftarrow 0 \end{aligned}$$

But C belongs to \mathcal{L} and E belongs to \mathcal{W} . Therefore $\text{Hom}(E, C)$ has no homology and the composite map:

$$\text{Hom}(C, C) \longrightarrow \text{Hom}(A, C) \xrightarrow{=} C$$

is an isomorphism in homology. Because of Lemma 2.0, $\text{Hom}(C, C)$ is a graded differential ring and $H_*(\text{Hom}(C, C))$ is a graded ring.

Suppose C' is another localization of A . Then there exist two morphisms: $f : C \rightarrow C'$ and $g : C' \rightarrow C$ of degree 0 compatible with the cofibrations $A \rightarrow C$ and $A \rightarrow C'$. These morphisms are unique up to homotopy and g is a homotopy inverse of f . So we have a morphism $\varphi : \text{Hom}(C, C) \rightarrow \text{Hom}(C', C')$ defined by:

$$\varphi(u) = f \circ u \circ g$$

Since f and g are homotopy inverses to each other, φ is a homotopy equivalence well defined up to homotopy. Moreover φ is compatible, up to homotopy, with the product and induces an isomorphism of rings between $H_*(\text{Hom}(C, C))$ and $H_*(\text{Hom}(C', C'))$. So the graded ring $\Lambda_* = H_*(\text{Hom}(C, C))$ is well defined up to canonical isomorphism.

Since C vanishes in negative degrees, $\Lambda_i = H_i(\text{Hom}(C, C)) \simeq H_i(C)$ is trivial for $i < 0$.

Let λ be the cofibration $A \rightarrow C$. Let a and b be two elements in A . These elements are two cycles in A and $\lambda(a)$ and $\lambda(b)$ are two cycles in C . These cycles define two morphisms from A to C sending 1 to $\lambda(a)$ and $\lambda(b)$ and can be extended to two endomorphisms $\theta(a)$ and $\theta(b)$ of degree 0 of C . So we have:

$$\theta(a)(\lambda(1)) = \lambda(a) \quad \theta(b)(\lambda(1)) = \lambda(b)$$

and that implies:

$$\theta(a) \circ \theta(b)(\lambda(1)) = \theta(a)(\lambda(b)) = \theta(a)(\lambda(1)b) = \theta(a)(\lambda(1))b = \lambda(a)b = \lambda(ab)$$

Therefore $\theta(ab)$ is homotopic to $\theta(a) \circ \theta(b)$ and the morphism from A to $H_0(C) \simeq H_0(\text{Hom}(C, C)) = \Lambda_0$ is compatible with the ring structure. Thus $A \rightarrow \Lambda_0$ is a ring homomorphism.

Let L be a local complex. We have the following exact sequence:

$$0 \longleftarrow \text{Hom}(A, L) \longleftarrow \text{Hom}(C, L) \longleftarrow \text{Hom}(E, L) \longleftarrow 0$$

and, as above, the restriction map $\text{Hom}(C, L) \rightarrow \text{Hom}(A, L) = L$ is an isomorphism in homology. But $\text{Hom}(C, C)$ acts on the right on $\text{Hom}(C, L)$ and we get a morphism from $\text{Hom}(C, L) \otimes \text{Hom}(C, C)$ to $\text{Hom}(C, L)$ inducing on $H_*(\text{Hom}(C, L)) = H_*(L)$ a structure of right Λ_* -module.

As above one checks that this structure is well defined and every morphism between two local complexes induces in homology a Λ_* -linear morphism.

Consider a complex C . Denote by C_p the component of C of degree p and by $C(p)$ the p -skeleton of C . The complexes $\Phi(C(p))$ define a filtration of $\Phi(C)$ and then a spectral sequence converging to the homology of $\Phi(C)$. The E^1 term of this spectral sequence is:

$$E_{pq}^1 = H_{p+q}(\Phi(C(p))/\Phi(C(p-1)))$$

Because of Proposition 2.6, we have:

$$E_{pq}^1 \simeq H_{p+q}(\Phi(C(p))/\Phi(C(p-1))) = H_{p+q}(\Phi(C_p)) \simeq H_{p+q}(C_p \otimes_A \Phi(A)) \simeq C_p \otimes_A \Lambda_q$$

Moreover the differential d^1 is induced by the differential of C . So we have:

$$E_{pq}^2 \simeq H_p(C \otimes_A \Lambda_q)$$

Now the last thing to do is to prove that Λ_0 is the Cohn localization of A with respect to W .

Let R be a ring. If M is a right R -module (resp. a left R -module), the dual module $D(M) = \text{Hom}(M, R)$ is a left R -module (resp. a right R -module). The correspondence D is a contravariant functor and, if M is finitely generated and projective, the module $D(M)$ is finitely generated and projective and the canonical morphism $M \rightarrow D^2(M)$ is an isomorphism.

Let $f : A \rightarrow R$ be a ring homomorphism. If M is a module in $\mathcal{P}(A)$, the right R -modules $D(\text{Hom}(M, R))$ and $M \otimes_A R$ are isomorphic. Therefore we have the following equivalences:

$$\begin{aligned} & \forall (P \rightarrow Q) \in W, \quad P \otimes_A R \xrightarrow{\sim} Q \otimes_A R \\ \iff & \forall (P \rightarrow Q) \in W, \quad D(P \otimes_A R) \xleftarrow{\sim} D(Q \otimes_A R) \\ \iff & \forall (P \rightarrow Q) \in W, \quad \text{Hom}(P, R) \xleftarrow{\sim} \text{Hom}(Q, R) \\ \iff & R \text{ is } W\text{-local as a right } A\text{-module} \end{aligned}$$

If this condition is satisfied consider a free resolution C of the right A -module R . The complex C is W -local and, because of the spectral sequence above, the morphism $R \rightarrow R \otimes_A \Lambda_0$ is an isomorphism. Then we have:

$$\begin{aligned} & \forall (P \rightarrow Q) \in W, \quad P \otimes_A R \xrightarrow{\sim} Q \otimes_A R \\ \iff & R \xrightarrow{\sim} R \otimes_A \Lambda_0 \end{aligned}$$

In particular the morphism $\Lambda \rightarrow \Lambda \otimes_A \Lambda_0$ is an isomorphism and Λ_0 has the following property:

$$\forall (P \rightarrow Q) \in W, \quad P \otimes_A \Lambda_0 \xrightarrow{\sim} Q \otimes_A \Lambda_0$$

Hence the ring homomorphism $\lambda_0 : A \rightarrow \Lambda_0$ factors through Λ :

$$A \xrightarrow{\lambda} \Lambda \xrightarrow{\mu} \Lambda_0$$

and we have a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & \Lambda \\ \lambda_0 \downarrow & \nearrow f & \downarrow \mu \\ \Lambda_0 & \xrightarrow{\text{Id}} & \Lambda_0 \end{array}$$

In this diagram, all the objects are rings and Λ and Λ_0 are right Λ_0 -modules. Morphisms λ and μ are ring homomorphisms and μ is Λ_0 -linear. The morphism f is the composite:

$$\Lambda_0 = A \otimes_A \Lambda_0 \xrightarrow{\lambda \otimes 1} \Lambda \otimes_A \Lambda_0 \simeq \Lambda$$

and f is Λ_0 -linear.

Thus $\mu : \Lambda \rightarrow \Lambda_0$ is a split surjection and we have a decomposition of right Λ_0 -modules: $\Lambda = \Lambda_0 \oplus J$. With this decomposition the maps f and μ are given by:

$$\forall (a, u) \in \Lambda_0 \times J, \quad f(a) = a \oplus 0, \quad \mu(a \oplus u) = a$$

Since μ is a ring homomorphism the product on Λ denoted by \times can be written this way:

$$(a \oplus u) \times (b \oplus v) = (ab \oplus \delta(a, b) + a \times v + u \times b + u \times v)$$

with a, b in Λ_0 and u, v in J . On the other hand the multiplication on the left by an element $x \in \Lambda$ corresponds to a morphism between W -local modules and this multiplication is a morphism of right Λ_0 -modules. So we have:

$$(a \oplus 0) \times (b \oplus 0) = (ab \oplus \delta(a, b)) = (a \oplus 0) \times (1 \oplus 0)b = ab \oplus \delta(a, 1)b \implies \delta(a, b) = \delta(a, 1)b$$

Since λ is a ring homomorphism, $1 \oplus 0$ is equal to 1 and that implies:

$$a \oplus 0 = (a \oplus 0) \times (1, 0) = a \oplus \delta(a, 1) \implies \delta(a, 1) = 0$$

Hence the map δ is trivial and f is a ring homomorphism.

Therefore $\Lambda_0 \simeq f(\Lambda_0)$ is a subring of Λ and every morphism in W is sent to an isomorphism in $\mathcal{M}(\Lambda_0)$. By the universal property, the ring Λ_0 is the Cohn localization $L(A, W) = \Lambda$ of A . \square

2.8 Corollary: Let A be a ring and W be a class of morphisms in $\mathcal{M}(A)$. Let Λ_* be the graded localization ring of A with respect to W and $\Lambda = \Lambda_0$ be the corresponding Cohn localization. Let $p > 0$ be an integer such that Λ_i vanishes for $0 < i < p$. Let M be a W -local right A -module. Then we have isomorphisms of right Λ -modules:

$$M \simeq M \otimes_A \Lambda$$

$$\forall i, 0 < i \leq p, \quad \text{Tor}_i^A(M, \Lambda) \simeq 0$$

$$\text{Tor}_{p+1}^A(M, \Lambda) \simeq M \otimes_A \Lambda_p \simeq M \otimes_\Lambda \Lambda_p$$

Moreover, if M is the module Λ , the isomorphism: $\text{Tor}_{p+1}^A(\Lambda, \Lambda) \simeq \Lambda_p$ is an isomorphism of Λ -bimodules.

Proof: Consider a projective resolution C of the right A -module M . Since C is local $\lambda_C : C \rightarrow \Phi(C)$ is a homotopy equivalence and the spectral sequence of Theorem 2.7 implies the desired result. \square

2.9 Remark: Because of this corollary, Λ_i vanish for all $i > 0$ if and only if Λ is stably flat over A . An example of non stably flat localization rings was found by Schofield [S] but theorem B implies a lot of explicit other examples.

A last result in this section is a description of classes \mathcal{W} and \mathcal{L} using only the ring $\Lambda = L(A, W)$:

2.10 Proposition: The class \mathcal{W} is the smallest class in $\mathcal{C}_*(A)$ satisfying the conditions of Lemma 2.2. Moreover for every C in $\mathcal{C}_*(A)$, one has the following characterizations:

$$C \in \mathcal{W} \iff H_*(C \otimes_A \Lambda) = 0$$

$$C \in \mathcal{L} \iff H_*(C) \xrightarrow{\sim} H_*(C) \otimes_A \Lambda$$

Proof: Let \mathcal{W}' be the smallest class in $\mathcal{C}_*(A)$ satisfying the conditions of Lemma 2.2. Let C be a complex in \mathcal{W} . By construction the complex $\Phi(C)/C$ lies in \mathcal{W}' . Since C belongs to \mathcal{W} , $\Phi(C)$ is contractible and belongs to \mathcal{W}' . Therefore C belongs to \mathcal{W}' too and \mathcal{W} is the class \mathcal{W}' .

Let C be a complex and C' be the complex $C \otimes_A \Lambda$. Suppose C' is not acyclic and let p be the smallest integer such that $H_p(C') \neq 0$. The spectral sequence of Theorem 2.7 implies:

$$H_i(\Phi(C)) \simeq \begin{cases} H_p(C') & \text{if } i = p \\ 0 & \text{if } i < p \end{cases}$$

If C lies in \mathcal{W} , $\Phi(C)$ is acyclic and $H_*(C')$ has to be zero. Conversely, if C' is acyclic, $\Phi(C)$ is acyclic too and C belongs to \mathcal{W} .

Let M be a local module and C a free resolution of M . Because of Lemma 2.1, C lies in \mathcal{L} and we have the following isomorphisms:

$$M \simeq H_0(C) \simeq H_0(\Phi(C)) \simeq H_0(C) \otimes_A \Lambda \simeq M \otimes_A \Lambda$$

If C is a local complex, each homology module of C is local and the map $H_*(C) \rightarrow H_*(C) \otimes_A \Lambda$ is an isomorphism.

Conversely, if $H_*(C)$ is isomorphic to $H_*(C) \otimes_A \Lambda$, each $H_i(C)$ is local and C belongs to \mathcal{L} . \square

3. Proof of the main theorems.

If G is a group, the localization ring $L(\mathbf{Z}[G] \xrightarrow{\varepsilon} \mathbf{Z})$, where ε is the augmentation map, will be denoted by $\Lambda(G)$.

3.1 Lemma: *Let G be a group and H_1 and H_2 be two finite subgroups of G of coprime orders. Suppose H_2 normalizes H_1 . Then for every x in H_1 and y in H_2 , $(1-x)(1-y)$ and $(1-y)(1-x)$ vanish in $\Lambda(G)$.*

Proof: Let n_1 and n_2 be the orders of H_1 and H_2 . Since these two numbers are coprime, there exist two integers a and b with: $an_1 + bn_2 = 1$. Consider the following elements in $\mathbf{Z}[G]$:

$$N_1 = \sum_{z \in H_1} z \quad \omega_1 = \sum_{z \in H_1} (1-z) = n_1 - N_1$$

$$N_2 = \sum_{z \in H_2} z \quad \omega_2 = \sum_{z \in H_2} (1-z) = n_2 - N_2$$

Since y normalizes H_1 , we have the following:

$$N_1x = xN_1 = N_1 \quad N_2y = yN_2 = N_2 \quad N_1y = yN_1$$

and then:

$$\omega_1(1-x) = (1-x)\omega_1 = n_1(1-x)$$

$$\omega_2(1-y) = (1-y)\omega_2 = n_2(1-y)$$

$$\omega_1y = y\omega_1$$

Let U be the element $1 - a\omega_1 - b\omega_2 \in \mathbf{Z}[G]$. This element is sent to 1 in \mathbf{Z} and becomes invertible in $\Lambda(G)$. Moreover U commutes with y . Thus we have:

$$(1-x)(1-y)U = (1-x)(1-a\omega_1-b\omega_2)(1-y) = (1-x)(1-an_1-bn_2)(1-y) = 0$$

$$U(1-y)(1-x) = (1-y)(1-a\omega_1-b\omega_2)(1-x) = (1-y)(1-an_1-bn_2)(1-x) = 0$$

Since U is invertible in $\Lambda(G)$, $(1-x)(1-y)$ and $(1-y)(1-x)$ vanish in $\Lambda(G)$. \square

3.2 Lemma: *Let G be a finite group. Then the image of G in $\Lambda(G)$ is nilpotent. Moreover for every $x, y \in G$ of coprime orders, $(1-x)(1-y)$ vanishes in $\Lambda(G)$.*

Proof: Let G' be the image of the map $G \rightarrow \Lambda(G)$. Because of Proposition 1.11, the ring Λ is the localization ring $L(\mathbf{Z}[G'] \rightarrow \mathbf{Z})$. Because of Corollary 1.8, G' doesn't contain any perfect subgroup and G' is solvable. Thus there is a filtration:

$$1 = H_0 \subset H_1 \subset \dots \subset H_n = G'$$

of G' such that for each $k < n$, H_k is a normal subgroup of H_{k+1} and H_{k+1}/H_k is a group of prime order.

Let k be an integer with $0 \leq k < n$. Suppose H_k is nilpotent. Let p be the order of H_{k+1}/H_k and S be a p -Sylow subgroup of H_{k+1} . The composite map $S \rightarrow H_{k+1} \rightarrow H_{k+1}/H_k$ is surjective.

Let q be any prime. Since H_k is nilpotent it has a unique q -Sylow subgroup S_q and H_k is isomorphic to the product of all the S_q 's. Actually the group S_p is the group $S \cap H_k$. Let $S' \subset H_k$ be the product of all the S_q 's with $q \neq p$, and z be an element of S . Since H_k is nilpotent, it has a unique q -Sylow subgroup and the conjugation by z sends S_q to itself. Therefore z lies in the normalizer of each S_q and then in the normalizer of S' .

Let x be an element of S' . Because of Lemma 3.1, $(1-x)(1-z)$ and $(1-z)(1-x)$ vanish in $\Lambda(G')$ and xz and zx have the same image in $\Lambda(G')$. But G' is included in $\Lambda(G')$ and we have: $xz = zx$. Therefore each element in S commute with each element in S' . Let $H \subset H_{k+1}$ be the product $S \times S'$. Since H contains each S_q , it contains H_k . On the other hand H contains the group S which surjects onto H_{k+1}/H_k . Therefore H is the group H_{k+1} and $H_{k+1} = S \times S'$ is nilpotent.

Thus each H_k is nilpotent by induction and G' is nilpotent too.

Let x and y be two elements in G of coprime orders a and b . Let x' and y' be the images of x and y in G' and H_1 (resp. H_2) be the product of all the q -Sylow subgroups of G' with q dividing a (resp. b). Then x' lies in H_1 and y' lies in H_2 . Since G' is nilpotent, each element in H_1 commutes with each element in H_2 and, by Lemma 3.1, $(1-x')(1-y')$ vanishes in $\Lambda(G') = \Lambda(G)$. Thus $(1-x)(1-y)$ vanishes in $\Lambda(G)$. \square

Actually there is a more general setting of this result:

3.3 Proposition: *Let $f : A \rightarrow B$ be a ring homomorphism and $\Lambda = L^h(A \rightarrow B)$ be the corresponding Cohn localization. Let G be a finite group contained in the group of units of A and sent to 1 in B . Then the image G' of G in Λ is nilpotent and, for every $x, y \in G$ of coprime orders, $(1-x)(1-y)$ vanishes in Λ . Moreover for each prime p invertible in B , the order of G' is coprime to p .*

Proof: We have two commutative diagrams of rings:

$$\begin{array}{ccc}
\mathbf{Z}[G] & \longrightarrow & \mathbf{Z} \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{Z}[G] & \longrightarrow & L(\mathbf{Z}[G] \rightarrow \mathbf{Z}) \\
\downarrow & & \downarrow \\
A & \longrightarrow & \Lambda
\end{array}$$

Since the image of G in $L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ is nilpotent by lemma 3.2, the image of G in Λ is also nilpotent. Moreover $(1-x)(1-y)$ vanishes in $L(\mathbf{Z}[G] \rightarrow \mathbf{Z})$ and then in Λ .

Suppose p is a prime invertible in B . Let S be a p -Sylow subgroup of G' . If S is non trivial, there is an element $x \in G'$ of order p . We have:

$$0 = 1 - x^p = (1 - x)U$$

with: $U = 1 + x + x^2 + \dots + x^{p-1}$. Then U is sent to p in B and U is invertible in Λ . Thus $1 - x$ vanishes in Λ and x is equal to 1. Hence S is trivial and the order of G' is coprime to p . \square

Let G be a finite group. For any prime p , denote by G_p the quotient of G by the subgroup generated by the elements of G of order coprime to p . The group G_p is the biggest p -group quotient of G . Denote by G' the product of the G_p 's. The morphism $G \rightarrow G'$ is surjective and G' is the universal nilpotent quotient of G . Because of Proposition 1.11 and Lemma 3.2, the map $L(\mathbf{Z}[G] \rightarrow \mathbf{Z}) \rightarrow L(\mathbf{Z}[G'] \rightarrow \mathbf{Z})$ is an isomorphism.

Denote by $A(G)$ the ring defined by the pull-back diagram:

$$\begin{array}{ccc}
A(G) & \longrightarrow & \prod_p \mathbf{Z}[G_p] \\
\downarrow & & \downarrow \Pi_p \varepsilon_p \\
\mathbf{Z} & \xrightarrow{\Delta} & \prod_p \mathbf{Z}
\end{array}$$

where ε_p is the augmentation map, Δ the diagonal map and the product is over all non trivial G_p . There is a canonical ring homomorphism from $\mathbf{Z}[G]$ to $A(G)$. Moreover this morphism factors through $\mathbf{Z}[G']$.

3.4 Lemma: *The morphism $\mathbf{Z}[G] \rightarrow A(G)$ is onto and every element in its kernel is killed in $\Lambda(G)$.*

Proof: Since $A(G) \rightarrow A(G')$ and $\Lambda(G) \rightarrow \Lambda(G')$ are isomorphisms, it is enough to prove the lemma when G is nilpotent. In this case G is the product of the groups G_p .

For each group Γ denote by $I(\Gamma)$ the augmentation ideal of $\mathbf{Z}[\Gamma]$. In order to prove that $\mathbf{Z}[G] \rightarrow A(G)$ is onto, it is enough to prove that the induced morphism $I(G) \rightarrow \text{Ker}(A(G) \rightarrow \mathbf{Z})$ is onto. But the kernel of $A(G) \rightarrow \mathbf{Z}$ is the product of the ideals $I(G_p)$. So it is enough to show that the map $I(G) \rightarrow \prod_p I(G_p)$ is onto.

Let q be a prime and u be an element in $I(G_q)$. The inclusion $G_q \subset G$ sends u to an element $v \in I(G)$. Let $w = (w_p)$ be the image of v in the product $\prod_p I(G_p)$. By construction, we have: $w_q = u$ and $w_p = 0$ for every $p \neq q$. Therefore the map $I(G) \rightarrow \prod_p I(G_p)$ is onto.

For each prime p , denote by I_p the image of $I(G_p)$ under the inclusion map $\mathbf{Z}[G_p] \subset \mathbf{Z}[G]$. Because of Lemma 3.1, $I_p I_q$ is killed in $\Lambda(G)$ for every $p \neq q$. So denote by $J(G)$ the two-sided ideal of $\mathbf{Z}[G]$ generated by all the $I_p I_q$, $p \neq q$, and by $B(G)$ the quotient $\mathbf{Z}[G]/J(G)$. The ideal $J(G)$ is killed in $\Lambda(G)$ and the map $\mathbf{Z}[G] \rightarrow \Lambda(G)$ factors through $B(G)$. By construction $J(G)$ is killed in $A(G)$ and the map $\mathbf{Z}[G] \rightarrow A(G)$ factors through $B(G)$. So, in order to prove the lemma, it is enough to prove that the induced map $f : B(G) \rightarrow A(G)$ is an isomorphism.

Sub-lemma: *Let x be an element of G . For each p denote by x_p the image of x under the projection $G \rightarrow G_p$. Then $1 - x - \sum_p (1 - x_p)$ vanishes in $B(G)$.*

Using this sub-lemma we are able to finish the proof of the lemma. Let $u \in \mathbf{Z}[G]$ be an element sent to 0 in $A(G)$. Then u is sent to 0 in \mathbf{Z} and u lies in $I(G)$. So we have a decomposition:

$$u = \sum_i \lambda_i (1 - x_i)$$

where the x_i 's are in G and the λ_i 's in \mathbf{Z} . Because of the sub-lemma, we may as well suppose that each x_i lies in some G_p and we have a decomposition: $u = \sum_p u_p$, where each u_p lies in $I(G_p)$.

Since u vanishes in $A(G)$, it vanishes in $\mathbf{Z}[G_p]$ for each p and u_p is zero for each p . Therefore u is sent to 0 in $B(G)$ and f is injective. So f is bijective and the lemma is proven. \square

Proof of the sub-lemma: Suppose x is the product $x = y_1 y_2 \dots y_k$ where y_i belongs to G_{p_i} , and all the primes p_i are distinct. Denote by \equiv the equality in $B(G)$. We have:

$$1 - x = 1 - y_1 y_2 \dots y_k = 1 - y_1 y_2 \dots y_{k-1} + y_1 y_2 \dots y_{k-1} (1 - y_k)$$

But we have for every $i < k$: $y_i (1 - y_k) \equiv 1 - y_k$. Therefore we have:

$$1 - x \equiv 1 - y_1 y_2 \dots y_{k-1} + (1 - y_k)$$

and, by induction, we get:

$$\begin{aligned} 1 - y_1 y_2 \dots y_k &\equiv 1 - y_1 y_2 \dots y_{k-1} + (1 - y_k) \equiv 1 - y_1 y_2 \dots y_{k-2} + (1 - y_{k-1}) + (1 - y_k) \\ &\equiv \dots \equiv (1 - y_1) + (1 - y_2) + \dots + (1 - y_k) \end{aligned}$$

The result follows. \square

For each prime p , set:

$$n_p = \text{card}(G_p) \quad N_p = \sum_{x \in G_p} x \quad \omega_p = n_p - N_p = \sum_{x \in G_p} (1 - x)$$

ω_p lies in the center of $\mathbf{Z}[G']$ and we have: $\omega_p^2 = n_p \omega_p$. Consider the multiplicative sets $S_p = 1 + \mathbf{Z}\omega_p$ and $\Sigma_p = 1 + \mathbf{Z}n_p$. Since n_p is a power of p , we have:

$$\Sigma_p^{-1} \mathbf{Z} = L^s(\mathbf{Z} \rightarrow \mathbf{Z}/n_p) = L^h(\mathbf{Z} \rightarrow \mathbf{Z}/n_p) = \mathbf{Z}_{(p)}$$

Let R be the subring of $\mathbf{Z}[G']$ generated by all the ω_p 's and J be the ideal of R generated by the ω_p 's. The ring R is contained in the center of $\mathbf{Z}[G']$ and R/J is the ring \mathbf{Z} . So we have another multiplicative set in the center of $\mathbf{Z}[G']$: the set $S = 1 + J$.

If D is a commutative square in a category of modules, we say that D is exact if D is cartesian and cocartesian.

3.5 Lemma: *Let p be a prime. Then we have an exact square of right $\mathbf{Z}[G]$ -modules:*

$$\begin{array}{ccc} S^{-1}\mathbf{Z}[G_p] & \longrightarrow & \mathbf{Z}_{(p)}[G_p] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}_{(p)} \end{array}$$

Proof: The two-sided ideal of $\mathbf{Z}[G_p]$ generated by N_p is the module $(N_p) = \mathbf{Z}N_p$. Hence the following square is exact:

$$\begin{array}{ccc} \mathbf{Z}[G_p] & \longrightarrow & \mathbf{Z}[G_p]/(N_p) \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/n_p \end{array}$$

Since ω_p is killed in \mathbf{Z} , we have:

$$S_p^{-1} \mathbf{Z} = \mathbf{Z} \quad S_p^{-1} \mathbf{Z}/n_p = \mathbf{Z}/n_p$$

But S_p is equal to Σ_p mod N_p . So we have:

$$S_p^{-1}(\mathbf{Z}[G_p]/(N_p)) = \Sigma_p^{-1}(\mathbf{Z}[G_p]/(N_p)) = \mathbf{Z}_{(p)}[G_p]/(N_p)$$

and we have an exact square:

$$\begin{array}{ccc} S_p^{-1}\mathbf{Z}[G_p] & \longrightarrow & \mathbf{Z}_{(p)}[G_p]/(N_p) \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}/n_p \end{array}$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc} S_p^{-1}\mathbf{Z}[G_p] & \longrightarrow & \mathbf{Z}_{(p)}[G_p] & \longrightarrow & \mathbf{Z}_{(p)}[G_p]/(N_p) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}_{(p)} & \longrightarrow & \mathbf{Z}/n_p \end{array}$$

In the square on the right the horizontal maps are surjective with isomorphic kernels. Then this square is exact. The composite square is also exact. Therefore the square on the left is exact. On the other hand ω_q is killed in $\mathbf{Z}[G_p]$ for every $q \neq p$. Then $S_p^{-1}\mathbf{Z}[G_p]$ is equal to the ring $S^{-1}\mathbf{Z}[G_p]$ and the lemma is proven. \square

3.6 Lemma: *There is an exact sequence of right $A(G)$ -modules:*

$$0 \longrightarrow \bigoplus_p I(G_p)_{(p)} \longrightarrow S^{-1}A(G) \longrightarrow \mathbf{Z} \longrightarrow 0$$

Proof: We have the following exact sequence:

$$0 \longrightarrow I(G_p) \longrightarrow \mathbf{Z}[G_p] \longrightarrow \mathbf{Z} \longrightarrow 0$$

But Lemma 3.5 implies the exact sequence:

$$0 \longrightarrow I(G_p)_{(p)} \longrightarrow S^{-1}\mathbf{Z}[G_p] \longrightarrow \mathbf{Z} \longrightarrow 0$$

and $S^{-1}I(G_p)$ is isomorphic to $I(G_p)_{(p)}$.

By construction we have an exact sequence:

$$0 \longrightarrow \bigoplus_p I(G_p) \longrightarrow A(G) \longrightarrow \mathbf{Z} \longrightarrow 0$$

and by inverting S we get the desired exact sequence. \square

Proof of Theorem A:

The pull-back diagram defining the ring $A(G)$ and the previous lemma imply that we have an exact square:

$$\begin{array}{ccc} S^{-1}A(G) & \longrightarrow & \prod_p \mathbf{Z}_{(p)}[G_p] \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \prod_p \mathbf{Z}_{(p)} \end{array}$$

Thus the only thing to do is to prove that $S^{-1}A(G)$ is the Cohn localization of the augmentation map $\mathbf{Z}[G] \rightarrow \mathbf{Z}$.

Since S may be seen as a multiplicative set in $A(G)$ sent to $1 \in \mathbf{Z}$, we have the following:

$$L(\mathbf{Z}[G] \rightarrow \mathbf{Z}) \simeq L(\mathbf{Z}[G'] \rightarrow \mathbf{Z}) \simeq L(A(G) \rightarrow \mathbf{Z}) \simeq L(S^{-1}A(G) \rightarrow \mathbf{Z})$$

Recall that a ring homomorphism f was called local if f is surjective and every element in $f^{-1}(1)$ is invertible. Because of Proposition 1.10, each morphism $\mathbf{Z}_{(p)}[G_p] \rightarrow \mathbf{Z}_{(p)}$ is local. Then the product of these maps is local and, because the square above is exact, the map $S^{-1}A(G) \rightarrow \mathbf{Z}$ is local too. Then the desired result follows from Lemma 1.9. \square

It is also possible to determine the ring $L(\mathbf{Z}[G] \rightarrow \mathbf{Z}/p)$. Actually one has the following:

3.7 Proposition: *Let G be a finite group and p be a prime. Then the localization ring $L(\mathbf{Z}[G] \rightarrow \mathbf{Z}_{(p)}) = L(\mathbf{Z}[G] \rightarrow \mathbf{Z}/p)$ is isomorphic to the ring $\mathbf{Z}_{(p)}[G_p]$.*

Proof: Because of Proposition 3.3 and Lemma 1.9, we have:

$$L(\mathbf{Z}[G] \rightarrow \mathbf{Z}/p) \simeq L(\mathbf{Z}[G_p] \rightarrow \mathbf{Z}/p)$$

Since each integer coprime to p is invertible in \mathbf{Z}/p , we have also:

$$L(\mathbf{Z}[G_p] \rightarrow \mathbf{Z}/p) \simeq L(\mathbf{Z}_{(p)}[G_p] \rightarrow \mathbf{Z}/p) \simeq L(\mathbf{Z}_{(p)}[G_p] \rightarrow \mathbf{Z}_{(p)})$$

and $L(\mathbf{Z}[G] \rightarrow \mathbf{Z}/p)$ is the ring $\mathbf{Z}_{(p)}[G_p]$ because of Proposition 1.10. \square

Actually it is possible to give a description of the graded localization ring of the augmentation morphism $\mathbf{Z}[G] \rightarrow \mathbf{Z}$. In order to do that we'll take the following notations, for each group G , each prime p and each pointed space X :

- $\Lambda(G)$ denote the localization ring of the augmentation morphism $\mathbf{Z}[G] \rightarrow \mathbf{Z}$ and $\Lambda(G)_*$ the corresponding graded localization ring.

- $\Lambda_{(p)}(G)$ denote the localization ring of the augmentation morphism $\mathbf{Z}[G] \rightarrow \mathbf{Z}_{(p)}$ and $\Lambda_{(p)}(G)_*$ the corresponding graded localization ring.
- BG denote the classifying space of G . This space is a pointed space.
- ΩX denote the loop space of X .
- $X_{(p)}$ denote the localization space of X with respect to the homology $H_*(-, \mathbf{Z}_{(p)})$.

3.8 Theorem: *Let G be a finite group. Then the graded localization ring $\Lambda(G)_*$ is determined by the following pull-back diagram of graded rings:*

$$\begin{array}{ccc} \Lambda(G)_* & \longrightarrow & \prod_p \Lambda_{(p)}(G)_* \\ \downarrow & & \downarrow \\ \mathbf{Z} & \longrightarrow & \prod_p \mathbf{Z}_{(p)} \end{array}$$

where the product is over all prime p dividing the order of G .

Moreover, for each prime p , $\Lambda_{(p)}(G)_*$ is isomorphic to $H_*(\Omega(BG_{(p)}), \mathbf{Z}_{(p)})$ as a graded ring.

Proof: Denote by \mathcal{W} the class of $\mathbf{Z}[G]$ -complexes C such that $H_*(C \otimes_{\mathbf{Z}[G]} \Lambda(G))$ vanishes and by \mathcal{L} the corresponding class of local complexes.

For each prime p , the class of $\mathbf{Z}[G]$ -complexes C such that $H_*(C \otimes_{\mathbf{Z}[G]} \Lambda_{(p)}(G))$ vanishes will be denoted by \mathcal{W}_p and the corresponding class of local complexes will be denoted by \mathcal{L}_p . We have the following inclusions:

$$\mathcal{W} \subset \mathcal{W}_p \quad \mathcal{L}_p \subset \mathcal{L}$$

Denote by H_p the kernel of the quotient map $G \rightarrow G_p$ and by F_p the homotopy fiber of the map $BG \rightarrow BG_{(p)}$. Since: $H_1(H_p, \mathbf{Z}_{(p)}) = 0$, the localization map $BG \rightarrow BG_{(p)}$ is characterized by the following properties:

$$\pi_1(BG_{(p)}) \simeq G_p$$

$$\forall i > 1, \quad \pi_i(BG_{(p)}) \text{ is a } \mathbf{Z}_{(p)}\text{-module}$$

$$\forall i > 0, \quad H_i(F_p, \mathbf{Z}_{(p)}) = 0$$

Let $\widetilde{BG_{(p)}}$ be the universal cover of $BG_{(p)}$, we have the following:

$$\forall i > 1, \quad \pi_i(BG_{(p)}) \text{ is a } \mathbf{Z}_{(p)}\text{-module}$$

$$\iff \forall i > 1, \quad \pi_i(\widetilde{BG_{(p)}}) \text{ is a } \mathbf{Z}_{(p)}\text{-module}$$

$$\iff \forall i > 0, \quad \pi_i(\Omega(\widetilde{BG_{(p)}})) \text{ is a } \mathbf{Z}_{(p)}\text{-module}$$

$$\iff \forall i > 0, \quad H_i(\Omega(\widetilde{BG_{(p)}}), \mathbf{Z}) \text{ is a } \mathbf{Z}_{(p)}\text{-module}$$

$$\iff \forall i > 0, \quad H_i(\Omega(BG_{(p)}), \mathbf{Z}) \text{ is a } \mathbf{Z}_{(p)}[G_p]\text{-module}$$

Let λ be the map: $F_p \rightarrow BG$. Since F_p is only defined up to homotopy, λ may be chosen to be a fibration. Denote by EG the universal cover of BG . The group G acts on the right on EG with a free action and BG is the quotient EG/G . Denote by \widehat{F}_p the space defined by the pull-back diagram:

$$\begin{array}{ccc} \widehat{F}_p & \xrightarrow{\widehat{\lambda}} & EG \\ \downarrow & & \downarrow \\ F_p & \xrightarrow{\lambda} & BG \end{array}$$

Actually, this diagram is a diagram of pointed spaces and the group G acts freely on \widehat{F}_p . Then the pull-back in \widehat{F}_p of the base point $x \in F_p$ is the discrete group G .

Let M_p be a \mathbf{Z} -complex which is a free resolution of $\mathbf{Z}_{(p)}$. The inclusion $\mathbf{Z} \subset \mathbf{Z}_{(p)}$ and the product on $\mathbf{Z}_{(p)}$ are induced by morphisms $\mathbf{Z} \rightarrow M_p$ and $M_p \otimes M_p \rightarrow M_p$. Let C_0 be the chain complex $C_*(G, \mathbf{Z})$ and C_1 be the chain complex $C_*(\widehat{F}_p, \mathbf{Z})$. Because of the free G -action, C_0 and C_1 are $\mathbf{Z}[G]$ -complexes. Moreover the inclusion $\mathbf{Z}[G] \subset C_0$ is a homotopy equivalence. We will prove that the morphism:

$$\mathbf{Z}[G] \subset C_0 \subset C_1 = C_1 \otimes \mathbf{Z} \rightarrow C_1 \otimes M_p$$

is a \mathscr{W}_p -localization.

We have:

$$H_*(C_1/C_0 \otimes_{\mathbf{Z}[G]} \mathbf{Z}_{(p)}[G_p]) = H_*(F_p, x; \mathbf{Z}_{(p)}[G_p]) = H_*(F_p, x; \mathbf{Z}_{(p)}) \otimes_{\mathbf{Z}_{(p)}} \mathbf{Z}_{(p)}[G_p] = 0$$

and, because of Proposition 2.10, $C_0 \rightarrow C_1$ is a \mathscr{W}_p -equivalence. But the morphism $C_1 \simeq C_1 \otimes \mathbf{Z} \rightarrow C_1 \otimes M_p$ is a \mathscr{W}_p -equivalence. Then the morphism $\mathbf{Z}[G] \rightarrow C_1 \otimes M_p$ is a \mathscr{W}_p -equivalence too.

On the other hand the fibers of λ and $\widehat{\lambda}$ are both homotopically equivalent to the space $\Omega(BG_{(p)})$ and \widehat{F}_p has the homotopy type of $\Omega(BG_{(p)})$. Thus the homology $H_*(C_1 \otimes M_p) \simeq H_*(\Omega(BG_{(p)}), \mathbf{Z}_{(p)})$ is a $\mathbf{Z}_{(p)}[G_p]$ -module. Because of Proposition 2.10, $C_1 \otimes M_p$ belongs to \mathscr{L}_p and $C_1 \otimes M_p$ is a \mathscr{W}_p -localization of C_0 and then of $\mathbf{Z}[G]$. Hence we have isomorphisms of $\mathbf{Z}[G]$ -modules:

$$H_*(C_1 \otimes M_p) \simeq \Lambda_{(p)}(G)_* \simeq H_*(\Omega(BG_{(p)}), \mathbf{Z}_{(p)})$$

Now we have to prove that the isomorphism $\Lambda_{(p)}(G)_* \simeq H_*(\Omega(BG_{(p)}), \mathbf{Z}_{(p)})$ is an isomorphism of graded rings.

Let C be the chain complex $C_*(\Omega(BG_{(p)}), \mathbf{Z})$ and C' be the complex $C \otimes M_p$. Then the morphism $\mathbf{Z}[G] \rightarrow C'$ is a \mathscr{W}_p -localization of $\mathbf{Z}[G]$. On C' there is two morphisms

μ_1, μ_2 from $C' \otimes_{\mathbf{Z}} C'$ to C' : the morphism μ_1 inducing on $\Lambda_{(p)}(G)_*$ the canonical structure of graded ring and the morphism μ_2 coming from the H-space structure of $\Omega(BG_{(p)})$ and the product $M_p \otimes M_p \rightarrow M_p$. Since the map $G \rightarrow \Omega(BG_{(p)})$ is a morphism of H-spaces, the morphisms:

$$C' \otimes_{\mathbf{Z}} \mathbf{Z}[G] \longrightarrow C' \otimes_{\mathbf{Z}} C' \xrightarrow{\mu_i} C'$$

are both homotopic to the composite:

$$C' \otimes_{\mathbf{Z}} \mathbf{Z}[G] \longrightarrow C' \otimes_{\mathbf{Z}[G]} \mathbf{Z}[G] \simeq C'$$

Let K be the cokernel of the morphism: $\mathbf{Z}[G] \longrightarrow C'$. This complex belongs to \mathcal{W}_p . The morphism $\mu_1 - \mu_2$ is homotopic to a composite:

$$C' \otimes_{\mathbf{Z}} C' \longrightarrow C' \otimes_{\mathbf{Z}} K \xrightarrow{\nu} C'$$

But we have:

$$\mathrm{Hom}_{\mathbf{Z}[G]}(C' \otimes_{\mathbf{Z}} K, C') \simeq \mathrm{Hom}_{\mathbf{Z}}(C', \mathrm{Hom}_{\mathbf{Z}[G]}(K, C'))$$

and ν can be seen as a morphism $\tilde{\nu} : C' \longrightarrow \mathrm{Hom}_{\mathbf{Z}_{(p)}[G]}(K, C')$. But K belongs to \mathcal{W}_p and C to \mathcal{L}_p . Then the graded differential module $\mathrm{Hom}_{\mathbf{Z}[G]}(K, C')$ is acyclic. Hence $\tilde{\nu}$ is a morphism from a graded differential free \mathbf{Z} -module to an acyclic graded differential \mathbf{Z} -module and $\tilde{\nu}$ and ν are homotopic to 0. Therefore μ_1 and μ_2 are homotopic and $\Lambda_{(p)}(G)_*$ is isomorphic to $H_*(\Omega(BG_{(p)}), \mathbf{Z}_{(p)})$ as a graded ring.

If p doesn't divide the order of G , G_p is the trivial group and $\Omega(BG_{(p)})$ is contractible. In this case $\Lambda_{(p)}(G)_*$ is the graded ring $\mathbf{Z}_{(p)}$.

Let \mathcal{P} be the set of primes dividing the order of G . Denote by C a \mathcal{W} -localization of $\mathbf{Z}[G]$ and, for each prime p in \mathcal{P} , denote by C_p a \mathcal{W}_p -localization of $\mathbf{Z}[G]$. The morphism $\mathbf{Z}[G] \rightarrow C_p$ is a cofibration.

If E is a right $\mathbf{Z}[G]$ -module, denote by E_* a free resolution of E . Since $H_0(C_p)$ is isomorphic to $\mathbf{Z}_{(p)}[G_p]$, the augmentation morphism $\mathbf{Z}_{(p)}[G_p] \rightarrow \mathbf{Z}_{(p)}$ can be represented by a morphism $\varepsilon_p : C_p \rightarrow \mathbf{Z}_{(p)*}$. Since C_p is only defined up to homotopy, we may as well suppose that ε_p is onto.

Let Δ be a morphism from \mathbf{Z}_* to $\bigoplus_{p \in \mathcal{P}} \mathbf{Z}_{(p)*}$ inducing on H_0 the diagonal inclusion $\mathbf{Z} \rightarrow \bigoplus_{p \in \mathcal{P}} \mathbf{Z}_{(p)}$ and C' be the complex defined by the pull-back diagram:

$$\begin{array}{ccc} C' & \longrightarrow & \bigoplus_{p \in \mathcal{P}} C_p \\ \downarrow & & \downarrow \oplus \varepsilon_p \\ \mathbf{Z}_* & \xrightarrow{\Delta} & \bigoplus_{p \in \mathcal{P}} \mathbf{Z}_{(p)*} \end{array}$$

Since all the complexes \mathbf{Z}_* , $\mathbf{Z}_{(p)*}$, C_p are in \mathcal{L} , C' belongs to \mathcal{L} too and we have a commutative diagram:

$$\begin{array}{ccccccc}
\mathbf{Z}[G] & \longrightarrow & C & \xrightarrow{\lambda} & C' & \longrightarrow & \bigoplus_{p \in \mathcal{P}} C_p \\
& & & & \downarrow & & \downarrow \oplus \varepsilon_p \\
& & & & \mathbf{Z}_* & \xrightarrow{\Delta} & \bigoplus_{p \in \mathcal{P}} \mathbf{Z}_{(p)*}
\end{array}$$

If p is a prime, denote by U_p a free \mathbf{Z} -resolution of \mathbf{Z}/p . For any $q \neq p$, we have:

$$H_*(C_q \otimes U_p) \simeq H_*(\Omega(BG_{(q)}), \mathbf{Z}_{(q)} \otimes \mathbf{Z}/p) = 0$$

$$H_*(\mathbf{Z}_{(q)} \otimes U_p) \simeq \mathbf{Z}_{(q)} \otimes \mathbf{Z}/p = 0$$

and that implies that $C' \otimes U_p \rightarrow C_p \otimes U_p$ is a homotopy equivalence. Then we have:

$$\begin{aligned}
H_*(C'/\mathbf{Z}[G] \otimes U_p \otimes_{\mathbf{Z}[G]} \mathbf{Z}) &\simeq H_*(C_p/\mathbf{Z}[G] \otimes U_p \otimes_{\mathbf{Z}[G]} \mathbf{Z}) \simeq H_*(C_p/\mathbf{Z}[G] \otimes_{\mathbf{Z}[G]} U_p) \\
&\simeq H_*(C_p/\mathbf{Z}[G] \otimes_{\mathbf{Z}[G]} (U_p)_{(p)}) \simeq H_*(C_p/\mathbf{Z}[G] \otimes_{\mathbf{Z}[G]} \mathbf{Z}_{(p)} \otimes (U_p)_{(p)}) = 0
\end{aligned}$$

and $\lambda \otimes U_p$ is a \mathcal{W} -equivalence.

Let K be the homotopy kernel of λ . The complex K belongs to \mathcal{L} and $K \otimes U_p$ belongs to \mathcal{W} for every prime p . Therefore the multiplication by a prime p is a homotopy equivalence from K to K and $H_*(K)$ is a $\mathbf{Q}[G]$ -module.

Because of Theorem A, the morphism $H_0(C) \rightarrow H_0(C')$ is an isomorphism. Then the last thing to do is to prove that $H_i(C)$ and $H_i(C')$ are torsion \mathbf{Z} -modules for all $i > 0$.

In order to do that denote by Λ the ring $\Lambda(G)$ or one of the rings $\Lambda_{(p)}(G)$. Consider the spectral sequence defined in Theorem 2.7 for a resolution C of $\Lambda \otimes \mathbf{Q}$. Since $\mathbf{Q}[G]$ is semisimple, this spectral sequence collapses and all the modules $\Lambda_i \otimes \mathbf{Q}$ vanish for $i > 0$. Thus, for every $i > 0$ and every prime p , $H_i(C \otimes \mathbf{Q})$ and $H_i(C_p \otimes \mathbf{Q})$ vanish. Hence $H_i(C)$ and $H_i(C')$ are torsion \mathbf{Z} -modules for all $i > 0$ and the module $H_i(K)$ is torsion for every i and K is acyclic. Thus C' is a \mathcal{W} -localization of $\mathbf{Z}[G]$ and the theorem is proven. \square

Proof of theorem B:

It is easy to check the implications:

$$1) \implies 2) \implies 3) \implies 4)$$

Suppose G is nilpotent. Then the central elements ω_p are well defined in $\mathbf{Z}[G]$ and the multiplicative set S also. Moreover S lies in the center of $\mathbf{Z}[G]$. Let p and q

be two distinct primes and x and y be two elements in G_p and G_q respectively. We have the following:

$$\omega_p(1-x) = n_p(1-x) \quad \omega_q(1-y) = n_q(1-y)$$

take two integers a and b with: $an_p + bn_q = 1$. Then we have:

$$(1 - a\omega_p - b\omega_q)(1-x)(1-y) = (1 - an_p - bn_q)(1-x)(1-y) = 0$$

and $(1-x)(1-y)$ is killed in $S^{-1}\mathbf{Z}[G]$. Therefore the ring $S^{-1}\mathbf{Z}[G]$ is the ring $S^{-1}(A(G)) = \Lambda$. Thus the localization is a central localization and 5) implies 1). Then the last thing to do is to prove: 4) implies 5).

Suppose Λ is stably flat over $\mathbf{Z}[G]$. Consider the left $\mathbf{Z}[G]$ -module $U = \Lambda \otimes_{\mathbf{Z}} \Lambda$ where the G -action on U is defined by:

$$g(u \otimes v) = ug^{-1} \otimes gv$$

for every $(g, u, v) \in G \times \Lambda \times \Lambda$. An elementary computation shows that $\text{Tor}_i^{\mathbf{Z}[G]}(\Lambda, \Lambda)$ is isomorphic to $H_i(G, U)$ and $H_i(G, U)$ vanishes for all $i > 0$.

For any prime p , one has also the $\mathbf{Z}[G]$ -module: $V_p = \mathbf{Z}_{(p)}[G_p] \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}[G_p]$ where the G -action is defined by:

$$g(u \otimes v) = ug^{-1} \otimes gv$$

for every $(g, u, v) \in G \times \mathbf{Z}_{(p)}[G_p] \times \mathbf{Z}_{(p)}[G_p]$.

For each prime p , denote by H_p the kernel of the quotient map $G \rightarrow G_p$. The intersection H of the H_p 's is the kernel of the map $G \rightarrow G' = \prod_p G_p$. Denote also by E_p the ring $\mathbf{Q}[G_p]/(N_p)$, where N_p is the sum of all elements in G_p .

Let p be a prime. We have: $H_i(G, U)_{(p)} = H_i(G, U_{(p)})$ and $H_i(G, U_{(p)})$ vanishes for $i > 0$. Because of theorem A, the ring homomorphism

$$\Lambda \longrightarrow \mathbf{Z}_{(p)}[G_p] \times \prod_{q \neq p} E_q$$

induces an isomorphism:

$$\Lambda_{(p)} \xrightarrow{\sim} \mathbf{Z}_{(p)}[G_p] \times \prod_{q \neq p} E_q$$

and the $\mathbf{Z}[G]$ -module $U_{(p)}$ contains V_p as a direct summand. Therefore $H_i(G, V_p)$ vanishes for every $i > 0$.

Let W be the left $\mathbf{Z}[G]$ -module $\mathbf{Z}_{(p)}[G_p]$ and W_0 be the module W equipped with a trivial G -action. The morphism $V_p \longrightarrow W \otimes W_0$ defined by $u \otimes v \mapsto v \otimes uv$ for every u and v in G_p , is an isomorphism of $\mathbf{Z}[G]$ -modules. Then we have for every $i > 0$:

$$\begin{aligned} 0 &\simeq H_i(G, V_p) \simeq H_i(G, W \otimes W_0) = H_i(G, W) \otimes W_0 \\ &\implies 0 \simeq H_i(G, W) \simeq H_i(H_p, \mathbf{Z}_{(p)}) \end{aligned}$$

Because of the next lemma the order of H_p is coprime to p . Therefore the order of H is coprime to every prime and H is the trivial group. Thus $G \simeq \prod_p G_p$ is nilpotent. \square

Lemma: Let G be a finite group and p be a prime. Suppose $H_i(G, \mathbf{Z}_{(p)}) = 0$ for all $i > 0$. Then the order of G is coprime to p .

Proof: Since $H_i(G, \mathbf{Z}_{(p)})$ vanishes for $i > 0$, the Krull dimension of the \mathbf{F}_p -algebra $H^*(G, \mathbf{F}_p)$ (or equivalently of its center) is zero. But this dimension is known to be the maximal integer n such that G contains an elementary abelian p -group $(\mathbf{Z}/p)^n$ (see Quillen [Q] or Adem [A]).

Since this dimension is zero, G doesn't contain any non-trivial elementary abelian p -group and the order of G is coprime to p . \square

Example 1: Let p be a prime and n be an integer coprime to p . Let G be a group of order pn containing a cyclic subgroup H of order n . Suppose one has: $H = [G, H]$. Denote by Λ the localization of the augmentation morphism $\mathbf{Z}[G] \rightarrow \mathbf{Z}$ and by Λ_* the associated graded localization ring. The group G_p is the quotient G/H and the ring Λ is the fiber product:

$$\Lambda = \mathbf{Z}_{(p)}[G_p] \times_{\mathbf{Z}_{(p)}} \mathbf{Z}$$

Moreover there is an element α of degree $2p - 2$ such that:

$$\Lambda_* = \Lambda[\alpha]/(n\alpha) = \Lambda \oplus (\mathbf{Z}/n)\alpha \oplus (\mathbf{Z}/n)\alpha^2 \oplus (\mathbf{Z}/n)\alpha^3 \oplus \dots$$

In the spectral sequence of 2.7, the element $\alpha \in \Lambda_{2p-2}$ corresponds to a generator of the module:

$$\begin{aligned} \mathrm{Tor}_{2p-1}^{\mathbf{Z}[G]}(\Lambda, \Lambda) &\simeq \mathrm{Ker}(H_{2p-1}(G) \rightarrow H_{2p-1}(G_p)) \simeq H_0(G_p, H_{2p-1}(H)) \\ &\simeq H_{2p-1}(H) \otimes_G \mathbf{Z} \simeq \mathbf{Z}/n \end{aligned}$$

Example 2: Let G be the alternating group A_5 and \tilde{G} be its universal central extension. These groups are the unique perfect groups of order 60 and 120. Denote by Λ_* and $\tilde{\Lambda}_*$ the graded localization rings of the augmentation maps $\mathbf{Z}[G] \rightarrow \mathbf{Z}$ and $\mathbf{Z}[\tilde{G}] \rightarrow \mathbf{Z}$. Then we have:

$$\begin{aligned} \tilde{\Lambda}_* &= \mathbf{Z}[x]/(120x) \\ \Lambda_* &= \mathbf{Z}[t, y]/(2t, t^2, 60y) \end{aligned}$$

with: $\partial^\circ t = 1$, $\partial^\circ x = \partial^\circ y = 2$. Moreover the morphism $\tilde{G} \rightarrow G$ induces a morphism from $\tilde{\Lambda}_*$ to Λ_* sending x to y .

Counter-example 1: Let H be a subgroup of a group G such that $H = [G, H]$. If G is finite, the group H is killed in any nilpotent quotient of G and the localization of $\mathbf{Z}[G] \rightarrow \mathbf{Z}$ is isomorphic to the localization of $\mathbf{Z}[G/H] \rightarrow \mathbf{Z}$. But this property is not true in general even if H is finite as will be shown in the following example:

Let p be an odd prime. Consider the group G generated by two elements x and t with the only relations: $x^p = 1$ and $tx = x^{-1}t$. The center Z of G is a free group generated by $y = t^2$ and G/Z is a dihedral group of order $2p$. Let $H \subset G$

be the subgroup generated by x . The group H is isomorphic to \mathbf{Z}/p and we have: $H = [G, H]$. The Cohn localization of the augmentation map $\mathbf{Z}[G] \rightarrow \mathbf{Z}$ will be denoted by Λ . We will prove that the morphism $\mathbf{Z}[G] \rightarrow \Lambda$ is injective and doesn't factors through $\mathbf{Z}[G/H]$.

Consider the field $F_1 = \mathbf{Q}(t)$ of rational fractions in t . The correspondence $x \mapsto 1$ and $t \mapsto t$ induces a morphism f_1 from $\mathbf{Z}[G]$ to F_1 . It is clear that F_1 is a local $\mathbf{Z}[G]$ -module.

Consider the field $F_2 = \mathbf{Q}[\zeta_p](y)$ of rational fractions in y with coefficients in the cyclotomic field $\mathbf{Q}[\zeta_p]$. The complex conjugation $\zeta_p \mapsto \zeta_p^{-1}$ induces an involution $u \mapsto \bar{u}$ in F_2 . Denote by F_3 the subfield of F_2 generated by y and $\zeta_p + \zeta_p^{-1}$. The field F_2 is a Galois extension of F_3 of degree 2. Set: $A = F_2 \oplus F_2 t$. This module is a left F_2 -vector space. We define a product on A by:

$$(a + bt)(c + dt) = ac + b\bar{d}y + (ad + b\bar{c})t$$

and A is a ring. The involution on F_2 extends to an anti-involution on A by setting: $\bar{t} = -t$. It is easy to see that $u + \bar{u}$ and $u\bar{u}$ are in F_3 for each $u \in A$. Moreover for each non zero $u \in A$, $u\bar{u}$ is non zero too. Thus each non zero element $u \in A$ is invertible with inverse:

$$u^{-1} = \bar{u}(u\bar{u})^{-1}$$

and A is a division ring. Actually A can be embedded in the skew field of quaternions.

The correspondence $x \mapsto \zeta_p$ and $t \mapsto t$ induces a morphism f_2 from $\mathbf{Z}[G]$ to A . As above A is a local $\mathbf{Z}[G]$ -module.

Using f_1 and f_2 we have a morphism $f : \mathbf{Z}[G] \rightarrow F_1 \times A$ which factors through Λ . But a straightforward computation shows that f is injective and the morphism $\mathbf{Z}[G] \rightarrow \Lambda$ is injective too.

Counter-example 2: Let G be a group and x and y be two elements in G with coprime orders. Denote by Λ the localization ring of the augmentation map $\mathbf{Z}[G] \rightarrow \mathbf{Z}$. If G is finite, x and y commute in Λ and $(1 - x)(1 - y)$ vanishes in Λ . But this property is not true in general if G is infinite.

Actually, if G is defined by the following presentation:

$$G = \langle x, y | x^2 = 1, y^3 = 1 \rangle$$

the elements $(1 - x)(1 - y)$ and $(1 - y)(1 - x)$ do not vanish in Λ and x and y do not commute in Λ .

In order to prove that we'll consider the following rings:

- The ring $A = \mathbf{Z} + \mathbf{Z}j \subset \mathbf{C}$ generated by a third root of unit $j = \zeta_3$.
- The polynomial ring $B = A[t]$ and the ring $B' = S^{-1}B$ where S is the multiplicative set $1 + (1 - t)B$.

The ideal in B generated by $(1 + t, 1 - j)$ is denoted by J .

- The fraction field K of B (and of B').
- The ring $R \subset M_2(K)$ defined by:

$$R = \mathbf{Z} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} (1 - t)B & B \\ (1 - t^2)B & J \end{pmatrix}$$

The map sending the unit matrix to 1 and all the matrices in $\begin{pmatrix} (1-t)B & B \\ (1-t^2)B & J \end{pmatrix}$ to 0, is a ring homomorphism $f : R \rightarrow \mathbf{Z}$.

The correspondence $x \mapsto \begin{pmatrix} t & 1 \\ 1-t^2 & -t \end{pmatrix}$ and $y \mapsto \begin{pmatrix} 1 & 0 \\ 0 & j \end{pmatrix}$ defines a ring homomorphism $\varphi : \mathbf{Z}[G] \rightarrow R$. Moreover the morphism $f \circ \varphi$ is the augmentation map $\mathbf{Z}[G] \rightarrow \mathbf{Z}$.

Denote by Λ' the localization ring of $f : R \rightarrow \mathbf{Z}$. By naturality we have a commutative diagram:

$$\begin{array}{ccccc} \mathbf{Z}[G] & \xrightarrow{\varphi} & R & \longrightarrow & M_2(K) \\ \downarrow & & \downarrow & & \\ \Lambda & \longrightarrow & \Lambda' & & \end{array}$$

Let U be a $n \times n$ matrix with entries in R sent by f to the unit matrix in $M_n(\mathbf{Z})$. This matrix may be seen as a matrix:

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta$ in $B_n = M_n(B)$ and:

$$\gamma \in (1-t^2)B_n \quad \alpha - 1 \in (1-t)B_n \quad \delta - 1 \in JB_n$$

Since α is congruent to 1 mod $(1-t)B_n$, its determinant belongs to S and α is invertible in $M_n(B')$. Then there exist a matrix $\hat{\alpha} \in M_n(B)$ and a polynomial $u \in B$ such that:

$$\alpha^{-1} = (1 + (1-t)u)^{-1} \hat{\alpha}$$

Thus we have in B' :

$$\det(U) = \det(\alpha) \det(\delta - \gamma \alpha^{-1} \beta) = \det(\alpha) (1 + (1-t)u)^{-n} \det(\delta(1 + (1-t)u) - \gamma \hat{\alpha} \beta)$$

We have four ring homomorphisms: $g_+, g_- : B \rightarrow A$, $h_3 : A \rightarrow \mathbf{F}_3$ and $h_4 : A \rightarrow \mathbf{F}_4$ defined by:

$$g_+ : t \mapsto 1 \quad g_- : t \mapsto -1 \quad h_3 : j \mapsto 1 \quad h_4 : 2 \mapsto 0$$

Since g_+ sends α to $1 \in M_n(A)$ and $1 + (1-t)u$ to $1 \in A$, $\det(\alpha)(1 + (1-t)u)^{-n}$ is non zero in B' . We have:

$$\begin{aligned} g_-(\delta(1 + (1-t)u) - \gamma \hat{\alpha} \beta) &= g_-(\delta(1 + (1-t)u)) = g_-(\delta)(1 + 2g_-(u)) \\ \implies g_-(\det(\delta(1 + (1-t)u) - \gamma \hat{\alpha} \beta)) &= g_-(\det(\delta))(1 + 2g_-(u))^n \end{aligned}$$

On the other hand $1 + 2g_-(u)$ (resp. $g_-(\delta)$) is sent to 1 by h_4 (resp. h_3). Therefore $1 + 2g_-(u)$ and $g_-(\det(\delta))$ are non zero in A and $\det(\delta(1 + (1 - t)u) - \gamma\hat{\alpha}\beta)$ is non zero in $B \subset B'$.

Thus $\det(U)$ is non zero in $B' \subset K$ and U is invertible in $M_{2n}(K) \simeq M_n(M_2(K))$. By the universal property, the ring homomorphism $R \rightarrow M_2(K)$ factors through Λ' and the ring homomorphism $\mathbf{Z}[G] \rightarrow M_2(K)$ factors through Λ .

A straightforward computation shows the following:

$$\varphi((1 - x)(1 - y)) = (1 - j) \begin{pmatrix} 0 & -1 \\ 0 & 1 + t \end{pmatrix}$$

$$\varphi((1 - y)(1 - x)) = (1 - j)(1 + t) \begin{pmatrix} 0 & 0 \\ t - 1 & 1 \end{pmatrix}$$

$$\varphi(xy - yx) = (1 - j) \begin{pmatrix} 0 & -1 \\ 1 - t^2 & 0 \end{pmatrix}$$

and, because these matrices are not zero in $M_2(K)$, the three elements $(1 - x)(1 - y)$, $(1 - y)(1 - x)$ and $xy - yx$ do not vanish in Λ .

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